

Problem 21

In each of Problems 16 through 25, find all eigenvalues and eigenvectors of the given matrix.

$$\begin{pmatrix} -3 & 3/4 \\ -5 & 1 \end{pmatrix}$$

Solution

The aim is to solve the eigenvalue problem,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x},$$

where \mathbf{A} is the given matrix. Bring $\lambda\mathbf{x}$ to the left side and combine the terms.

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

Evaluate the determinant and solve for λ .

$$\det \begin{pmatrix} -3 - \lambda & 3/4 \\ -5 & 1 - \lambda \end{pmatrix} = 0$$

$$(-3 - \lambda)(1 - \lambda) + \frac{15}{4} = 0$$

$$\lambda^2 + 2\lambda + \frac{3}{4} = 0$$

$$\left(\lambda + \frac{3}{2}\right) \left(\lambda + \frac{1}{2}\right) = 0$$

Therefore, the eigenvalues are

$$\lambda_1 = -\frac{3}{2} \quad \text{and} \quad \lambda_2 = -\frac{1}{2}.$$

Substitute λ_1 and λ_2 back into equation (1) to determine the corresponding eigenvectors, \mathbf{x}_1 and \mathbf{x}_2 .

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x}_1 = \mathbf{0}$$

$$\begin{pmatrix} -3/2 & 3/4 \\ -5 & 5/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left. \begin{aligned} -1.5x_1 + 0.75x_2 &= 0 \\ -5x_1 + 2.5x_2 &= 0 \end{aligned} \right\}$$

$$x_2 = 2x_1$$

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x}_2 = \mathbf{0}$$

$$\begin{pmatrix} -5/2 & 3/4 \\ -5 & 3/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left. \begin{aligned} -2.5x_1 + 0.75x_2 &= 0 \\ -5x_1 + 1.5x_2 &= 0 \end{aligned} \right\}$$

$$x_2 = \frac{10}{3}x_1$$

$$\mathbf{x}_2 = \begin{pmatrix} x_1 \\ \frac{10}{3}x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ \frac{10}{3} \end{pmatrix}$$

Note that x_1 is a free variable, or arbitrary constant.