Problem 22

In each of Problems 16 through 25, find all eigenvalues and eigenvectors of the given matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$$

Solution

The aim is to solve the eigenvalue problem,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x},$$

where $\mathbf{A}$ is the given matrix. Bring $\lambda\mathbf{x}$ to the left side and combine the terms.

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

Evaluate the determinant and solve for λ.

$$\det\begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(1-\lambda)(1-\lambda)+4] = 0$$

$$5 - 7\lambda + 3\lambda^2 - \lambda^3 = 0$$

$$(1-\lambda)(\lambda^2 - 2\lambda + 5) = 0$$

$$1-\lambda = 0 \quad \text{or} \quad \lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = 1 \quad \text{or} \quad \lambda = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

Therefore, the eigenvalues are

$$\boxed{\lambda_1 = 1} \quad \text{and} \quad \boxed{\lambda_2 = 1 - 2i} \quad \text{and} \quad \boxed{\lambda_3 = 1 + 2i.}$$

Substitute λ_1 back into equation (1) to determine the corresponding eigenvector $\mathbf{x}_1$.

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x}_1 = \mathbf{0}$$

$$\begin{pmatrix} 1-(1) & 0 & 0 \\ 2 & 1-(1) & -2 \\ 3 & 2 & 1-(1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Write the implied system of equations.

$$\left.\begin{aligned} 2x_1 - 2x_3 &= 0 \\ 3x_1 + 2x_2 &= 0 \end{aligned}\right\}$$

Solve for x_2 and x_3 in terms of the free variable x_1.

$$x_3 = x_1$$

$$x_2 = -\frac{3}{2}x_1$$

This means

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -\frac{3}{2}x_1 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{pmatrix}.$$

Since x_1 is arbitrary, the eigenvector can be multiplied by 2 to get rid of the fraction.

$$\boxed{\mathbf{x}_1 = x_1' \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}}$$

Substitute λ_2 back into equation (1) to determine the corresponding eigenvector $\mathbf{x}_2$.

$$(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{x}_2 = \mathbf{0}$$

$$\begin{pmatrix} 1-(1-2i) & 0 & 0 \\ 2 & 1-(1-2i) & -2 \\ 3 & 2 & 1-(1-2i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2i & 0 & 0 \\ 2 & 2i & -2 \\ 3 & 2 & 2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Write the implied system of equations.

$$\left.\begin{aligned} 2ix_1 = 0 \\ 2x_1 + 2ix_2 - 2x_3 = 0 \\ 3x_1 + 2x_2 + 2ix_3 = 0 \end{aligned}\right\}$$

Solve for x_3 in terms of the free variable x_2.

$$x_1 = 0$$
$$x_3 = ix_2$$

This means

$$\mathbf{x}_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \\ ix_2 \end{pmatrix}.$$

Therefore,

$$\boxed{\mathbf{x}_2 = x_2 \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}.}$$

Substitute λ_3 back into equation (1) to determine the corresponding eigenvector $\mathbf{x}_3$.

$$(\mathbf{A} - \lambda_3 \mathbf{I})\mathbf{x}_3 = \mathbf{0}$$

$$\begin{pmatrix} 1-(1+2i) & 0 & 0 \\ 2 & 1-(1+2i) & -2 \\ 3 & 2 & 1-(1+2i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Write the implied system of equations.

$$\left.\begin{aligned} -2ix_1 &= 0 \\ 2x_1 - 2ix_2 - 2x_3 &= 0 \\ 3x_1 + 2x_2 - 2ix_3 &= 0 \end{aligned}\right\}$$

Solve for x_3 in terms of the free variable x_2.

$$\begin{aligned} x_1 &= 0 \\ x_3 &= -ix_2 \end{aligned}$$

This means

$$\mathbf{x}_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \\ -ix_2 \end{pmatrix}.$$

Therefore,

$$\boxed{\mathbf{x}_3 = x_2 \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}.}$$