Problem 23

In each of Problems 16 through 25, find all eigenvalues and eigenvectors of the given matrix.

$$\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$$

Solution

The aim is to solve the eigenvalue problem,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x},$$

where $\mathbf{A}$ is the given matrix. Bring $\lambda\mathbf{x}$ to the left side and combine the terms.

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

Evaluate the determinant and solve for λ.

$$\det\begin{pmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{pmatrix} = 0$$

$$(3-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ -4 & -1-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ -2 & -1-\lambda \end{vmatrix} + 2\begin{vmatrix} 1 & 4-\lambda \\ -2 & -4 \end{vmatrix} = 0$$

$$(3-\lambda)[(4-\lambda)(-1-\lambda)+4] - 2[1(-1-\lambda)+2] + 2[1(-4)+2(4-\lambda)] = 0$$

$$6 - 11\lambda + 6\lambda^2 - \lambda^3 = 0$$

$$(\lambda-1)(\lambda-2)(3-\lambda) = 0$$

Therefore, the eigenvalues are

$$\boxed{\lambda_1 = 1} \quad \text{and} \quad \boxed{\lambda_2 = 2} \quad \text{and} \quad \boxed{\lambda_3 = 3.}$$

Substitute λ_1 back into equation (1) to determine the corresponding eigenvector $\mathbf{x}_1$.

$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{x}_1 = \mathbf{0}$$

$$\begin{pmatrix} 3-(1) & 2 & 2 \\ 1 & 4-(1) & 1 \\ -2 & -4 & -1-(1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Write the implied system of equations.

$$\left.\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ x_1 + 3x_2 + x_3 &= 0 \\ -2x_1 - 4x_2 - 2x_3 &= 0 \end{aligned}\right\}$$

Add the first row to the third row.

$$\left.\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ x_1 + 3x_2 + x_3 &= 0 \\ -2x_2 &= 0 \end{aligned}\right\}$$

Solve for x_2 and x_3 in terms of the free variable x_1.

$$x_3 = -x_1$$

$$x_2 = 0$$

This means

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ -x_1 \end{pmatrix}.$$

Therefore,

$$\boxed{\mathbf{x}_1 = x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.}$$

Substitute λ_2 back into equation (1) to determine the corresponding eigenvector $\mathbf{x}_2$.

$$(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{x}_2 = \mathbf{0}$$

$$\begin{pmatrix} 3-(2) & 2 & 2 \\ 1 & 4-(2) & 1 \\ -2 & -4 & -1-(2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -4 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Write the implied system of equations.

$$\left.\begin{aligned} x_1 + 2x_2 + 2x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 0 \\ -2x_1 - 4x_2 - 3x_3 &= 0 \end{aligned}\right\}$$

Multiply the first row by -1 and add it to the second row.

$$\left.\begin{aligned} x_1 + 2x_2 + 2x_3 &= 0 \\ -x_3 &= 0 \\ -2x_1 - 4x_2 - 3x_3 &= 0 \end{aligned}\right\}$$

Solve for x_1 and x_3 in terms of the free variable x_2.

$$x_1 = -2x_2$$

$$x_3 = 0$$

This means

$$\mathbf{x}_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ x_2 \\ 0 \end{pmatrix}.$$

Therefore,

$$\boxed{\mathbf{x}_2 = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.}$$

Substitute λ_3 back into equation (1) to determine the corresponding eigenvector $\mathbf{x}_3$.

$$(\mathbf{A} - \lambda_3\mathbf{I})\mathbf{x}_3 = \mathbf{0}$$

$$\begin{pmatrix} 3-(3) & 2 & 2 \\ 1 & 4-(3) & 1 \\ -2 & -4 & -1-(3) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ -2 & -4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Write the implied system of equations.

$$\left.\begin{aligned} 2x_2 + 2x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \\ -2x_1 - 4x_2 - 4x_3 &= 0 \end{aligned}\right\}$$

Multiply the first row by 2 and add it to the third row.

$$\left.\begin{aligned} 2x_2 + 2x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \\ -2x_1 &= 0 \end{aligned}\right\}$$

Solve for x_1 and x_2 in terms of the free variable x_3.

$$x_2 = -x_3$$

$$x_1 = 0$$

This means

$$\mathbf{x}_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -x_3 \\ x_3 \end{pmatrix}.$$

Therefore,

$$\boxed{\mathbf{x}_3 = x_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.}$$