Problem 24

In each of Problems 16 through 25, find all eigenvalues and eigenvectors of the given matrix.

$$\begin{pmatrix} 11/9 & -2/9 & 8/9 \\ -2/9 & 2/9 & 10/9 \\ 8/9 & 10/9 & 5/9 \end{pmatrix}$$

Solution

The aim is to solve the eigenvalue problem,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x},$$

where $\mathbf{A}$ is the given matrix. Bring $\lambda\mathbf{x}$ to the left side and combine the terms.

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

Evaluate the determinant and solve for λ.

$$\det\begin{pmatrix} 11/9-\lambda & -2/9 & 8/9 \\ -2/9 & 2/9-\lambda & 10/9 \\ 8/9 & 10/9 & 5/9-\lambda \end{pmatrix} = 0$$

$$(11/9-\lambda)\begin{vmatrix} 2/9-\lambda & 10/9 \\ 10/9 & 5/9-\lambda \end{vmatrix} - (-2/9)\begin{vmatrix} -2/9 & 10/9 \\ 8/9 & 5/9-\lambda \end{vmatrix} + (8/9)\begin{vmatrix} -2/9 & 2/9-\lambda \\ 8/9 & 10/9 \end{vmatrix} = 0$$

$$\begin{aligned} (11/9-\lambda)[(2/9-\lambda)(5/9-\lambda) - 100/81] + (2/9)[(-2/9)(5/9-\lambda) - 80/81]& \\ + (8/9)[-20/81 - (8/9)(2/9-\lambda)] &= 0 \end{aligned}$$

$$-2 + \lambda + 2\lambda^2 - \lambda^3 = 0$$

$$(\lambda+1)(\lambda-1)(2-\lambda) = 0$$

Therefore, the eigenvalues are

$$\boxed{\lambda_1 = 1} \quad \text{and} \quad \boxed{\lambda_2 = 2} \quad \text{and} \quad \boxed{\lambda_3 = -1.}$$

Substitute λ_1 back into equation (1) to determine the corresponding eigenvector $\mathbf{x}_1$.

$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{x}_1 = \mathbf{0}$$

$$\begin{pmatrix} 11/9-(1) & -2/9 & 8/9 \\ -2/9 & 2/9-(1) & 10/9 \\ 8/9 & 10/9 & 5/9-(1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2/9 & -2/9 & 8/9 \\ -2/9 & -7/9 & 10/9 \\ 8/9 & 10/9 & -4/9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Write the augmented matrix.

$$\left(\begin{array}{ccc|c} 2/9 & -2/9 & 8/9 & 0 \\ -2/9 & -7/9 & 10/9 & 0 \\ 8/9 & 10/9 & -4/9 & 0 \end{array}\right)$$

Multiply each row by 9.

$$\left(\begin{array}{ccc|c} 2 & -2 & 8 & 0 \\ -2 & -7 & 10 & 0 \\ 8 & 10 & -4 & 0 \end{array}\right)$$

Multiply the first row by -4 and add it to the third row.

$$\left(\begin{array}{ccc|c} 2 & -2 & 8 & 0 \\ -2 & -7 & 10 & 0 \\ 0 & 18 & -36 & 0 \end{array}\right)$$

Add the first row to the second row.

$$\left(\begin{array}{ccc|c} 2 & -2 & 8 & 0 \\ 0 & -9 & 18 & 0 \\ 0 & 18 & -36 & 0 \end{array}\right)$$

Write the implied system of equations and solve for x_1 and x_2 in terms of the free variable x_3.

$$\left.\begin{aligned} 2x_1 - 2x_2 + 8x_3 &= 0 \\ -9x_2 + 18x_3 &= 0 \\ 18x_2 - 36x_3 &= 0 \end{aligned}\right\} \quad \rightarrow \quad \begin{aligned} x_1 &= -2x_3 \\ x_2 &= 2x_3 \end{aligned}$$

This means

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ 2x_3 \\ x_3 \end{pmatrix}.$$

Therefore,

$$\boxed{\mathbf{x}_1 = x_3 \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}.}$$

Substitute λ_2 back into equation (1) to determine the corresponding eigenvector $\mathbf{x}_2$.

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x}_2 = \mathbf{0}$$

$$\begin{pmatrix} 11/9-(2) & -2/9 & 8/9 \\ -2/9 & 2/9-(2) & 10/9 \\ 8/9 & 10/9 & 5/9-(2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -7/9 & -2/9 & 8/9 \\ -2/9 & -16/9 & 10/9 \\ 8/9 & 10/9 & -13/9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Write the augmented matrix.

$$\left(\begin{array}{ccc|c} -7/9 & -2/9 & 8/9 & 0 \\ -2/9 & -16/9 & 10/9 & 0 \\ 8/9 & 10/9 & -13/9 & 0 \end{array}\right)$$

Multiply each row by 9.

$$\left(\begin{array}{ccc|c} -7 & -2 & 8 & 0 \\ -2 & -16 & 10 & 0 \\ 8 & 10 & -13 & 0 \end{array}\right)$$

Multiply the second row by 4 and add it to the third row.

$$\left(\begin{array}{ccc|c} -7 & -2 & 8 & 0 \\ -2 & -16 & 10 & 0 \\ 0 & -54 & 27 & 0 \end{array}\right)$$

Multiply the first row by -8 and add it to the second row.

$$\left(\begin{array}{ccc|c} -7 & -2 & 8 & 0 \\ 54 & 0 & -54 & 0 \\ 0 & -54 & 27 & 0 \end{array}\right)$$

Write the implied system of equations and solve for x_1 and x_2 in terms of the free variable x_3.

$$\left.\begin{aligned} -7x_1 - 2x_2 + 8x_3 &= 0 \\ 54x_1 - 54x_3 &= 0 \\ -54x_2 + 27x_3 &= 0 \end{aligned}\right\} \quad \rightarrow \quad \begin{aligned} x_1 &= x_3 \\ x_2 &= \frac{1}{2}x_3 \end{aligned}$$

This means

$$\mathbf{x}_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ \frac{1}{2}x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix}.$$

Since x_3 is arbitrary, the eigenvector can be multiplied by 2 to get rid of the fraction.

$$\boxed{\mathbf{x}_2 = x_3' \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}}$$

Substitute λ_3 back into equation (1) to determine the corresponding eigenvector $\mathbf{x}_3$.

$$(\mathbf{A} - \lambda_3 \mathbf{I})\mathbf{x}_3 = \mathbf{0}$$

$$\begin{pmatrix} 11/9 - (-1) & -2/9 & 8/9 \\ -2/9 & 2/9 - (-1) & 10/9 \\ 8/9 & 10/9 & 5/9 - (-1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 20/9 & -2/9 & 8/9 \\ -2/9 & 11/9 & 10/9 \\ 8/9 & 10/9 & 14/9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Write the augmented matrix.

$$\left(\begin{array}{ccc|c} 20/9 & -2/9 & 8/9 & 0 \\ -2/9 & 11/9 & 10/9 & 0 \\ 8/9 & 10/9 & 14/9 & 0 \end{array}\right)$$

Multiply each row by 9.

$$\left(\begin{array}{ccc|c} 20 & -2 & 8 & 0 \\ -2 & 11 & 10 & 0 \\ 8 & 10 & 14 & 0 \end{array}\right)$$

Multiply the second row by 4 and add it to the third row.

$$\left(\begin{array}{ccc|c} 20 & -2 & 8 & 0 \\ -2 & 11 & 10 & 0 \\ 0 & 54 & 54 & 0 \end{array}\right)$$

Multiply the second row by 10 and add it to the first row.

$$\left(\begin{array}{ccc|c} 0 & 108 & 108 & 0 \\ -2 & 11 & 10 & 0 \\ 0 & 54 & 54 & 0 \end{array}\right)$$

Write the implied system of equations and solve for x_1 and x_3 in terms of the free variable x_2.

$$\left.\begin{aligned} 108x_2 + 108x_3 = 0 \\ -2x_1 + 11x_2 + 10x_3 = 0 \\ 54x_2 + 54x_3 = 0 \end{aligned}\right\} \quad \rightarrow \quad \begin{aligned} x_1 &= \frac{1}{2}x_2 \\ x_3 &= -x_2 \end{aligned}$$

This means

$$\mathbf{x}_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_2 \\ x_2 \\ -x_2 \end{pmatrix} = x_2 \begin{pmatrix} \frac{1}{2} \\ 1 \\ -1 \end{pmatrix}.$$

Since x_2 is arbitrary, the eigenvector can be multiplied by 2 to get rid of the fraction.

$$\boxed{\mathbf{x}_3 = x_2' \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}$$