Problem 25

In each of Problems 16 through 25, find all eigenvalues and eigenvectors of the given matrix.

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

Solution

The aim is to solve the eigenvalue problem,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x},$$

where $\mathbf{A}$ is the given matrix. Bring $\lambda\mathbf{x}$ to the left side and combine the terms.

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

Evaluate the determinant and solve for λ.

$$\det\begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix} = 0$$

$$(3-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & 3-\lambda \end{vmatrix} + 4\begin{vmatrix} 2 & -\lambda \\ 4 & 2 \end{vmatrix} = 0$$

$$(3-\lambda)[(-\lambda)(3-\lambda) - 4] - 2[2(3-\lambda) - 8] + 4(4 + 4\lambda) = 0$$

$$8 + 15\lambda + 6\lambda^2 - \lambda^3 = 0$$

$$(\lambda + 1)^2(8 - \lambda) = 0$$

Therefore, the eigenvalues are

$$\boxed{\lambda_1 = -1} \quad \text{and} \quad \boxed{\lambda_2 = 8.}$$

λ_1 has multiplicity 2, so two vectors are expected to be associated with it.

Substitute λ_1 back into equation (1) to determine the corresponding eigenvector $\mathbf{x}_1$.

$$(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{x}_1=\mathbf{0}$$

$$\begin{pmatrix} 3-(-1) & 2 & 4 \\ 2 & -(-1) & 2 \\ 4 & 2 & 3-(-1) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Write the augmented matrix.

$$\left(\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right)$$

Multiply the second row by -2 and add it to the first row.

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right)$$

Multiply the second row by -2 and add it to the third row.

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Write the implied system of equations and solve for x_2 in terms of the free variables, x_1 and x_3.

$$2x_1+x_2+2x_3=0 \quad \rightarrow \quad x_2=-2x_1-2x_3$$

This means

$$\mathbf{x}_1=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} x_1 \\ -2x_1-2x_3 \\ x_3 \end{pmatrix}=\begin{pmatrix} x_1 \\ -2x_1 \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ -2x_3 \\ x_3 \end{pmatrix}.$$

Therefore,

$$\boxed{\mathbf{x}_1=x_1\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}+x_3\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.}$$

We can check that this eigenvector is correct.

$$\begin{aligned}\mathbf{A}\mathbf{x}_1=\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ -2x_1-2x_3 \\ x_3 \end{pmatrix}=\begin{pmatrix} 3x_1+2(-2x_1-2x_3)+4x_3 \\ 2x_1+2x_3 \\ 4x_1+2(-2x_1-2x_3)+3x_3 \end{pmatrix}=\begin{pmatrix} -x_1 \\ 2x_1+2x_3 \\ -x_3 \end{pmatrix}&=(-1)\begin{pmatrix} x_1 \\ -2x_1-2x_3 \\ x_3 \end{pmatrix}\\ &=\lambda_1\mathbf{x}_1\end{aligned}$$

Substitute λ_2 back into equation (1) to determine the corresponding eigenvector $\mathbf{x}_2$.

$$(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{x}_2 = \mathbf{0}$$

$$\begin{pmatrix} 3-(8) & 2 & 4 \\ 2 & -(8) & 2 \\ 4 & 2 & 3-(8) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Write the augmented matrix.

$$\left(\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right)$$

Multiply the second row by -2 and add it to the third row.

$$\left(\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 0 & 18 & -9 & 0 \end{array}\right)$$

Divide the second row by 2.

$$\left(\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 18 & -9 & 0 \end{array}\right)$$

Multiply the second row by 5 and add it to the first row.

$$\left(\begin{array}{ccc|c} 0 & -18 & 9 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 18 & -9 & 0 \end{array}\right)$$

Write the implied system of equations and solve for x_1 and x_3 in terms of the free variable x_2.

$$\left.\begin{aligned} x_1 - 4x_2 + x_3 &= 0 \\ 18x_2 - 9x_3 &= 0 \end{aligned}\right\} \quad \rightarrow \quad \begin{aligned} x_1 &= 2x_2 \\ x_3 &= 2x_2 \end{aligned}$$

This means

$$\mathbf{x}_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_2 \\ 2x_2 \end{pmatrix}.$$

Therefore,

$$\boxed{\mathbf{x}_2 = x_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.}$$