

Problem 3

In each of Problems 1 through 6, either solve the given system of equations, or else show that there is no solution.

$$\begin{aligned}x_1 + 2x_2 - x_3 &= 2 \\2x_1 + x_2 + x_3 &= 1 \\x_1 - x_2 + 2x_3 &= -1\end{aligned}$$

Solution

Start by calculating the determinant of the coefficient matrix.

$$\begin{aligned}\det \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} &= 1 \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 1 \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \\ &= 1(2 + 1) - 2(4 - 1) - 1(-2 - 1) \\ &= 0\end{aligned}$$

Since it's zero, either there is no solution to the system of equations or there are many.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & -1 \end{array} \right)$$

Multiply the first row by -2 and add it to the second row.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 1 & -1 & 2 & -1 \end{array} \right)$$

Multiply the first row by -1 and add it to the third row.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & -3 & 3 & -3 \end{array} \right)$$

Add the second row to the third row.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Divide the second row by 3.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

The first two rows imply that

$$\begin{aligned}x_1 + 2x_2 - x_3 &= 2 \\-x_2 + x_3 &= -1.\end{aligned}$$

Solve the second equation for x_2 .

$$x_2 = 1 + x_3$$

Substitute this formula into the first equation.

$$x_1 + 2(1 + x_3) - x_3 = 2$$

Solve for x_1 .

$$x_1 = -x_3$$

In terms of the free variable, $x_3 = c$, the solution to the system of equations is

$$\{-c, 1 + c, c\}.$$