Problem 4

In each of Problems 1 through 6, either solve the given system of equations, or else show that there is no solution.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$

Solution

Start by calculating the determinant of the coefficient matrix.

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} &= 1 \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} - 2 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - 1 \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \\ &= 1(2+1) - 2(4-1) - 1(-2-1) \\ &= 0 \end{aligned}$$

Since it's zero, either there is no solution to the system of equations or there are many.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right)$$

Multiply the first row by -2 and add it to the second row.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right)$$

Multiply the first row by -1 and add it to the third row.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right)$$

Add the second row to the third row.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Divide the second row by 3.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The first two rows imply that

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + x_3 &= 0. \end{aligned}$$

Solve the second equation for x_2.

$$x_2 = x_3$$

Substitute this formula into the first equation.

$$x_1 + 2(x_3) - x_3 = 0$$

Solve for x_1.

$$x_1 = -x_3$$

In terms of the free variable, $x_3 = c$, the solution to the system of equations is

$$\{-c, c, c\}.$$