Problem 4

In each of Problems 1 through 6, either solve the given system of equations, or else show that there is no solution.

\[
\begin{align*}
x_1 + 2x_2 - x_3 &= 0 \\
2x_1 + x_2 + x_3 &= 0 \\
x_1 - x_2 + 2x_3 &= 0
\end{align*}
\]

Solution

Start by calculating the determinant of the coefficient matrix.

\[
\begin{vmatrix}
1 & 2 & -1 \\
2 & 1 & 1 \\
1 & -1 & 2
\end{vmatrix}
= 1 \begin{vmatrix}
1 & 1 \\
-1 & 2
\end{vmatrix} - 2 \begin{vmatrix}
2 & 1 \\
1 & 2
\end{vmatrix} - 1 \begin{vmatrix}
2 & 1 \\
1 & -1
\end{vmatrix}
= (2 + 1) - 2(4 - 1) - 1(-2 - 1)
= 0
\]

Since it’s zero, either there is no solution to the system of equations or there are many.

\[
\begin{pmatrix}
1 & 2 & -1 & 0 \\
2 & 1 & 1 & 0 \\
1 & -1 & 2 & 0
\end{pmatrix}
\]

Multiply the first row by \(-2\) and add it to the second row.

\[
\begin{pmatrix}
1 & 2 & -1 & 0 \\
0 & -3 & 3 & 0 \\
1 & -1 & 2 & 0
\end{pmatrix}
\]

Multiply the first row by \(-1\) and add it to the third row.

\[
\begin{pmatrix}
1 & 2 & -1 & 0 \\
0 & -3 & 3 & 0 \\
0 & -3 & 3 & 0
\end{pmatrix}
\]

Add the second row to the third row.

\[
\begin{pmatrix}
1 & 2 & -1 & 0 \\
0 & -3 & 3 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Divide the second row by 3.

\[
\begin{pmatrix}
1 & 2 & -1 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

The first two rows imply that

\[
\begin{align*}
x_1 + 2x_2 - x_3 &= 0 \\
-x_2 + x_3 &= 0
\end{align*}
\]
Solve the second equation for $x_2$.

$$x_2 = x_3$$

Substitute this formula into the first equation.

$$x_1 + 2(x_3) - x_3 = 0$$

Solve for $x_1$.

$$x_1 = -x_3$$

In terms of the free variable, $x_3 = c$, the solution to the system of equations is

$$\{-c, c, c\}.$$