Problem 6

In each of Problems 1 through 6, either solve the given system of equations, or else show that there is no solution.

\[
\begin{align*}
  x_1 + 2x_2 - x_3 &= -2 \\
  -2x_1 - 4x_2 + 2x_3 &= 4 \\
  2x_1 + 4x_2 - 2x_3 &= -4 \\
\end{align*}
\]

Solution

Start by calculating the determinant of the coefficient matrix.

\[
\det \begin{pmatrix}
  1 & 2 & -1 \\
  -2 & -4 & 2 \\
  2 & 4 & -2 \\
\end{pmatrix} = 1 \left( \begin{array}{c}
  -4 & 2 \\
  4 & -2 \\
\end{array} \right) - 2 \left( \begin{array}{c}
  -2 & 2 \\
  2 & -2 \\
\end{array} \right) - 1 \left( \begin{array}{c}
  -2 & -4 \\
  2 & 4 \\
\end{array} \right)
\]

\[
= 1(8 - 8) - 2(4 - 4) - 1(-8 + 8) = 0
\]

Since it’s zero, either there is no solution to the system of equations or there are many.

\[
\begin{pmatrix}
  1 & 2 & -1 & -2 \\
  -2 & -4 & 2 & 4 \\
  2 & 4 & -2 & -4 \\
\end{pmatrix}
\]

Add the second row to the third row.

\[
\begin{pmatrix}
  1 & 2 & -1 & -2 \\
  -2 & -4 & 2 & 4 \\
  0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Multiply the first row by 2 and add it to the second row.

\[
\begin{pmatrix}
  1 & 2 & -1 & -2 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

The first row implies that

\[
x_1 + 2x_2 - x_3 = -2.
\]

Solve for \(x_3\).

\[
x_3 = 2 + x_1 + 2x_2.
\]

If we set \(x_1 = c_1\) and \(x_2 = c_2\) to be free variables, then the solution to the system of equations is

\[
\{c_1, c_2, 2 + c_1 + 2c_2\}.
\]