Problem 6

In each of Problems 1 through 6, either solve the given system of equations, or else show that there is no solution.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -2 \\ -2x_1 - 4x_2 + 2x_3 &= 4 \\ 2x_1 + 4x_2 - 2x_3 &= -4 \end{aligned}$$

Solution

Start by calculating the determinant of the coefficient matrix.

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & -1 \\ -2 & -4 & 2 \\ 2 & 4 & -2 \end{pmatrix} &= 1 \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} - 2 \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} - 1 \begin{pmatrix} -2 & -4 \\ 2 & 4 \end{pmatrix} \\ &= 1(8-8) - 2(4-4) - 1(-8+8) \\ &= 0 \end{aligned}$$

Since it's zero, either there is no solution to the system of equations or there are many.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -2 \\ -2 & -4 & 2 & 4 \\ 2 & 4 & -2 & -4 \end{array}\right)$$

Add the second row to the third row.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -2 \\ -2 & -4 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Multiply the first row by 2 and add it to the second row.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The first row implies that

$$x_1 + 2x_2 - x_3 = -2.$$

Solve for x_3.

$$x_3 = 2 + x_1 + 2x_2.$$

If we set $x_1 = c_1$ and $x_2 = c_2$ to be free variables, then the solution to the system of equations is

$$\{c_1, c_2, 2 + c_1 + 2c_2\}.$$