Problem 8

In each of Problems 7 through 11, determine whether the members of the given set of vectors are linearly independent. If they are linearly dependent, find a linear relation among them. The vectors are written as row vectors to save space but may be considered as column vectors; that is, the transposes of the given vectors may be used instead of the vectors themselves.

$$\mathbf{x}^{(1)} = (2, 1, 0), \qquad \mathbf{x}^{(2)} = (0, 1, 0), \qquad \mathbf{x}^{(3)} = (-1, 2, 0)$$

Solution

The three vectors are linearly dependent if there exists a nontrivial solution to

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$$

for c_1, c_2, and c_3. Rewrite this equation.

$$c_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Calculate the determinant of the coefficient matrix.

$$\det \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = 0 \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} - 0 \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} + 0 \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = 0$$

Since it's zero, there are infinitely many solutions for c_1, c_2, and c_3.

$$2c_1 - c_3 = 0$$
$$c_1 + c_2 + 2c_3 = 0$$

Solve this first equation for c_3

$$c_3 = 2c_1$$

and plug it into the second one.

$$c_1 + c_2 + 2(2c_1) = 0$$

Solve for c_2.

$$c_2 = -5c_1$$

In terms of the free variable c_1, the solution to the system of equations is

$$\{c_1, -5c_1, 2c_1\}.$$

For example, choose $c_1 = 1$. Then

$$\mathbf{x}^{(1)} - 5\mathbf{x}^{(2)} + 2\mathbf{x}^{(3)} = \mathbf{0}.$$

Therefore, the three given vectors are linearly dependent.