Problem 9

In each of Problems 7 through 11, determine whether the members of the given set of vectors are linearly independent. If they are linearly dependent, find a linear relation among them. The vectors are written as row vectors to save space but may be considered as column vectors; that is, the transposes of the given vectors may be used instead of the vectors themselves.

$$\mathbf{x}^{(1)} = (1, 2, 2, 3), \qquad \mathbf{x}^{(2)} = (-1, 0, 3, 1), \qquad \mathbf{x}^{(3)} = (-2, -1, 1, 0), \qquad \mathbf{x}^{(4)} = (-3, 0, -1, 3)$$

Solution

The four vectors are linearly dependent if there exists a nontrivial solution to

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} + c_4\mathbf{x}^{(4)} = \mathbf{0}$$

for c_1, c_2, c_3, and c_4. Rewrite this equation.

$$c_1\begin{pmatrix}1\\2\\2\\3\end{pmatrix} + c_2\begin{pmatrix}-1\\0\\3\\1\end{pmatrix} + c_3\begin{pmatrix}-2\\-1\\1\\0\end{pmatrix} + c_4\begin{pmatrix}-3\\0\\-1\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & -1 & -2 & -3\\2 & 0 & -1 & 0\\2 & 3 & 1 & -1\\3 & 1 & 0 & 3\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Calculate the determinant of the coefficient matrix.

$$\det\begin{pmatrix}1 & -1 & -2 & -3\\2 & 0 & -1 & 0\\2 & 3 & 1 & -1\\3 & 1 & 0 & 3\end{pmatrix} = -2\begin{vmatrix}-1 & -2 & -3\\3 & 1 & -1\\1 & 0 & 3\end{vmatrix} - (-1)\begin{vmatrix}1 & -1 & -3\\2 & 3 & -1\\3 & 1 & 3\end{vmatrix} = 0$$

Since it's zero, there are infinitely many solutions for c_1, c_2, c_3, and c_4.

$$\left(\begin{array}{cccc|c}1 & -1 & -2 & -3 & 0\\2 & 0 & -1 & 0 & 0\\2 & 3 & 1 & -1 & 0\\3 & 1 & 0 & 3 & 0\end{array}\right)$$

Multiply the first row by -2 and add it to the second row.

$$\left(\begin{array}{cccc|c}1 & -1 & -2 & -3 & 0\\0 & 2 & 3 & 6 & 0\\2 & 3 & 1 & -1 & 0\\3 & 1 & 0 & 3 & 0\end{array}\right)$$

Multiply the first row by -2 and add it to the third row.

$$\left(\begin{array}{cccc|c}1 & -1 & -2 & -3 & 0\\0 & 2 & 3 & 6 & 0\\0 & 5 & 5 & 5 & 0\\3 & 1 & 0 & 3 & 0\end{array}\right)$$

Multiply the first row by -3 and add it to the fourth row.

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & -3 & 0 \\ 0 & 2 & 3 & 6 & 0 \\ 0 & 5 & 5 & 5 & 0 \\ 0 & 4 & 6 & 12 & 0 \end{array}\right)$$

Multiply the second row by -2 and add it to the fourth row.

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & -3 & 0 \\ 0 & 2 & 3 & 6 & 0 \\ 0 & 5 & 5 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Divide the third row by 5.

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & -3 & 0 \\ 0 & 2 & 3 & 6 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Add the third row to the first row.

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 2 & 3 & 6 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Multiply the third row by -2 and add it to the second row.

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Multiply the second row by -1 and add it to the third row.

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

These first three rows imply that

$$\begin{aligned} c_1 - c_3 - 2c_4 &= 0 \\ c_3 + 4c_4 &= 0 \\ c_2 - 3c_4 &= 0. \end{aligned}$$

Solve for c_1, c_2, and c_3 in terms of the free variable c_4.

$$\begin{aligned} c_1 &= -2c_4 \\ c_2 &= 3c_4 \\ c_3 &= -4c_4 \end{aligned}$$

The solution to the system of equations is then

$$\{-2c_4, 3c_4, -4c_4, c_4\}.$$

For example, choose $c_4 = 1$. Then

$$-2\mathbf{x}^{(1)} + 3\mathbf{x}^{(2)} - 4\mathbf{x}^{(3)} + \mathbf{x}^{(4)} = \mathbf{0}.$$

Therefore, the four given vectors are linearly dependent.