

Problem 10

In each of Problems 9 through 14, find the general solution of the given system of equations.

$$\mathbf{x}' = \begin{pmatrix} 2 & 2+i \\ -1 & -1-i \end{pmatrix} \mathbf{x}$$

Solution

Because this is a constant-coefficient homogeneous linear system, it's expected to have solutions of the form $\mathbf{x} = e^{\lambda t} \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ has constant elements.

$$\lambda e^{\lambda t} \boldsymbol{\xi} = \begin{pmatrix} 2 & 2+i \\ -1 & -1-i \end{pmatrix} e^{\lambda t} \boldsymbol{\xi}$$

Divide both sides by $e^{\lambda t}$.

$$\lambda \boldsymbol{\xi} = \begin{pmatrix} 2 & 2+i \\ -1 & -1-i \end{pmatrix} \boldsymbol{\xi}$$

This is now an eigenvalue problem.

$$\left\{ \begin{pmatrix} 2 & 2+i \\ -1 & -1-i \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right\} \boldsymbol{\xi} = \mathbf{0}$$

$$\begin{pmatrix} 2-\lambda & 2+i \\ -1 & -1-i-\lambda \end{pmatrix} \boldsymbol{\xi} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det \begin{pmatrix} 2-\lambda & 2+i \\ -1 & -1-i-\lambda \end{pmatrix} = 0.$$

Evaluate the determinant and solve for λ .

$$\begin{aligned} (2-\lambda)(-1-i-\lambda) + 2+i &= 0 \\ -i - (1-i)\lambda + \lambda^2 &= 0 \\ (\lambda+i)(\lambda-1) &= 0 \\ \lambda &= \{-i, 1\} \end{aligned}$$

Let

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = -i.$$

Substitute these two eigenvalues into equation (1) to determine the corresponding eigenvectors.

$$\begin{aligned} \begin{pmatrix} 2-\lambda_1 & 2+i \\ -1 & -1-i-\lambda_1 \end{pmatrix} \boldsymbol{\xi}_1 &= \mathbf{0} & \begin{pmatrix} 2-\lambda_2 & 2+i \\ -1 & -1-i-\lambda_2 \end{pmatrix} \boldsymbol{\xi}_2 &= \mathbf{0} \\ \begin{pmatrix} 1 & 2+i \\ -1 & -2-i \end{pmatrix} \boldsymbol{\xi}_1 &= \mathbf{0} & \begin{pmatrix} 2+i & 2+i \\ -1 & -1 \end{pmatrix} \boldsymbol{\xi}_2 &= \mathbf{0} \\ \xi_1 + (2+i)\xi_2 &= 0 & (2+i)\xi_1 + (2+i)\xi_2 &= 0 \\ \xi_1 &= -(2+i)\xi_2 & \xi_1 &= -\xi_2 \\ \boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -(2+i)\xi_2 \\ \xi_2 \end{pmatrix} = \xi_2 \begin{pmatrix} -2-i \\ 1 \end{pmatrix} & & \boldsymbol{\xi}_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -\xi_2 \\ \xi_2 \end{pmatrix} = \xi_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{aligned}$$

Two solutions to the system are then

$$\mathbf{x}_1 = e^{\lambda_1 t} \begin{pmatrix} -2 - i \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{\lambda_2 t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Therefore, by the principle of superposition, the general solution is

$$\mathbf{x} = C_1 e^t \begin{pmatrix} -2 - i \\ 1 \end{pmatrix} + C_2 e^{-it} \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

where C_1 and C_2 are arbitrary constants.