Problem 11

In each of Problems 9 through 14, find the general solution of the given system of equations.

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \mathbf{x}$$

Solution

Because this is a constant-coefficient homogeneous linear system, it's expected to have solutions of the form $\mathbf{x} = e^{\lambda t}\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ has constant elements.

$$\lambda e^{\lambda t}\boldsymbol{\xi} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} e^{\lambda t}\boldsymbol{\xi}$$

Divide both sides by $e^{\lambda t}$.

$$\lambda \boldsymbol{\xi} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \boldsymbol{\xi}$$

This is now an eigenvalue problem.

$$\left\{ \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right\} \boldsymbol{\xi} = \mathbf{0}$$

$$\begin{pmatrix} 1-\lambda & 1 & 2 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{pmatrix} \boldsymbol{\xi} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det \begin{pmatrix} 1-\lambda & 1 & 2 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{pmatrix} = 0.$$

Evaluate the determinant and solve for λ.

$$(1-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 2 & 1-\lambda \end{vmatrix} + 2 \begin{vmatrix} 1 & 2-\lambda \\ 2 & 1 \end{vmatrix} = 0$$

$$(1-\lambda)[(2-\lambda)(1-\lambda) - 1] - 1[1(1-\lambda) - 2] + 2[(1)(1) - 2(2-\lambda)] = 0$$

$$-4 + \lambda + 4\lambda^2 - \lambda^3 = 0$$

$$(\lambda+1)(\lambda-1)(4-\lambda) = 0$$

$$\lambda = \{-1, 1, 4\}$$

Let

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = -1 \quad \text{and} \quad \lambda_3 = 4.$$

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Substitute these two eigenvalues into equation (1) to determine the corresponding eigenvectors.

$$\begin{pmatrix} 1-\lambda_1 & 1 & 2 \\ 1 & 2-\lambda_1 & 1 \\ 2 & 1 & 1-\lambda_1 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0} \qquad \begin{pmatrix} 1-\lambda_2 & 1 & 2 \\ 1 & 2-\lambda_2 & 1 \\ 2 & 1 & 1-\lambda_2 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0} \qquad \begin{pmatrix} 1-\lambda_3 & 1 & 2 \\ 1 & 2-\lambda_3 & 1 \\ 2 & 1 & 1-\lambda_3 \end{pmatrix} \boldsymbol{\xi}_3 = \mathbf{0}$$

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0} \qquad \begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0} \qquad \begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} \boldsymbol{\xi}_3 = \mathbf{0}$$

$$\left.\begin{aligned} \xi_2 + 2\xi_3 &= 0 \\ \xi_1 + \xi_2 + \xi_3 &= 0 \\ 2\xi_1 + \xi_2 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} 2\xi_1 + \xi_2 + 2\xi_3 &= 0 \\ \xi_1 + 3\xi_2 + \xi_3 &= 0 \\ 2\xi_1 + \xi_2 + 2\xi_3 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} -3\xi_1 + \xi_2 + 2\xi_3 &= 0 \\ \xi_1 - 2\xi_2 + \xi_3 &= 0 \\ 2\xi_1 + \xi_2 - 3\xi_3 &= 0 \end{aligned}\right\}$$

$$\xi_3 = -\frac{1}{2}\xi_2 \qquad \xi_1 = -\frac{1}{2}\xi_2 \qquad\qquad \xi_2 = 0 \qquad \xi_1 = -\xi_3 \qquad\qquad \xi_3 = \xi_2 \qquad \xi_1 = \xi_2$$

$$\boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\xi_2 \\ \xi_2 \\ -\frac{1}{2}\xi_2 \end{pmatrix} = \xi_2' \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \qquad \boldsymbol{\xi}_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} -\xi_3 \\ 0 \\ \xi_3 \end{pmatrix} = \xi_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad \boldsymbol{\xi}_3 = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_2 \\ \xi_2 \\ \xi_2 \end{pmatrix} = \xi_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Three solutions to the system are then

$$\mathbf{x}_1 = e^{\lambda_1 t} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{\lambda_2 t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_3 = e^{\lambda_3 t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Therefore, by the principle of superposition, the general solution is

$$\mathbf{x} = C_1 e^{t} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + C_3 e^{4t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

where C_1 and C_2 and C_3 are arbitrary constants.