Problem 12

In each of Problems 9 through 14, find the general solution of the given system of equations.

$$\mathbf{x}' = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \mathbf{x}$$

Solution

Because this is a constant-coefficient homogeneous linear system, it's expected to have solutions of the form $\mathbf{x} = e^{\lambda t}\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ has constant elements.

$$\lambda e^{\lambda t}\boldsymbol{\xi} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} e^{\lambda t}\boldsymbol{\xi}$$

Divide both sides by $e^{\lambda t}$.

$$\lambda\boldsymbol{\xi} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \boldsymbol{\xi}$$

This is now an eigenvalue problem.

$$\left\{ \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right\} \boldsymbol{\xi} = \mathbf{0}$$

$$\begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix} \boldsymbol{\xi} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det \begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix} = 0.$$

Evaluate the determinant and solve for λ.

$$(3-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & 3-\lambda \end{vmatrix} + 4\begin{vmatrix} 2 & -\lambda \\ 4 & 2 \end{vmatrix} = 0$$

$$(3-\lambda)[(-\lambda)(3-\lambda) - 4] - 2[2(3-\lambda) - 8] + 4[(2)(2) + 4\lambda] = 0$$

$$8 + 15\lambda + 6\lambda^2 - \lambda^3 = 0$$

$$(\lambda+1)^2(8-\lambda) = 0$$

$$\lambda = \{-1, 8\}$$

Let

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = 8.$$

Substitute these two eigenvalues into equation (1) to determine the corresponding eigenvectors.

$$\begin{pmatrix} 3-\lambda_1 & 2 & 4 \\ 2 & -\lambda_1 & 2 \\ 4 & 2 & 3-\lambda_1 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0} \qquad \begin{pmatrix} 3-\lambda_2 & 2 & 4 \\ 2 & -\lambda_2 & 2 \\ 4 & 2 & 3-\lambda_2 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0}$$

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0} \qquad \begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0}$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right)$$

$$\xi_2 = -2\xi_1 - 2\xi_3 \qquad \xi_1 = \xi_3 \qquad \xi_2 = \frac{1}{2}\xi_3$$

$$\boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ -2\xi_1 - 2\xi_3 \\ \xi_3 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \xi_3 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \qquad \boldsymbol{\xi}_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_3 \\ \frac{1}{2}\xi_3 \\ \xi_3 \end{pmatrix} = \xi_3' \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

Three solutions to the system are then

$$\mathbf{x}_1 = e^{\lambda_1 t} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{\lambda_1 t} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_3 = e^{\lambda_2 t} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

Therefore, by the principle of superposition, the general solution is

$$\mathbf{x} = e^{-t} \left\{ C_1 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\} + C_3 e^{8t} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix},$$

where C_1 and C_2 and C_3 are arbitrary constants.