Problem 14

In each of Problems 9 through 14, find the general solution of the given system of equations.

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \mathbf{x}$$

Solution

Because this is a constant-coefficient homogeneous linear system, it's expected to have solutions of the form $\mathbf{x} = e^{\lambda t}\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ has constant elements.

$$\lambda e^{\lambda t}\boldsymbol{\xi} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} e^{\lambda t}\boldsymbol{\xi}$$

Divide both sides by $e^{\lambda t}$.

$$\lambda\boldsymbol{\xi} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \boldsymbol{\xi}$$

This is now an eigenvalue problem.

$$\left\{ \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right\} \boldsymbol{\xi} = \mathbf{0}$$

$$\begin{pmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{pmatrix} \boldsymbol{\xi} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det \begin{pmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{pmatrix} = 0.$$

Evaluate the determinant and solve for λ.

$$(1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} + 1\begin{vmatrix} 3 & -1 \\ 2 & -1-\lambda \end{vmatrix} + 4\begin{vmatrix} 3 & 2-\lambda \\ 2 & 1 \end{vmatrix} = 0$$

$$(1-\lambda)[(2-\lambda)(-1-\lambda)+1] + 1[3(-1-\lambda)+2] + 4[(1)(3) - 2(2-\lambda)] = 0$$

$$-6 + 5\lambda + 2\lambda^2 - \lambda^3 = 0$$

$$(\lambda+2)(\lambda-1)(3-\lambda) = 0$$

$$\lambda = \{-2, 1, 3\}$$

Let

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = -2 \quad \text{and} \quad \lambda_3 = 3.$$

Substitute these three eigenvalues into equation (1) to determine the corresponding eigenvectors.

$$\begin{pmatrix} 1-\lambda_1 & -1 & 4 \\ 3 & 2-\lambda_1 & -1 \\ 2 & 1 & -1-\lambda_1 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0} \qquad \begin{pmatrix} 1-\lambda_2 & -1 & 4 \\ 3 & 2-\lambda_2 & -1 \\ 2 & 1 & -1-\lambda_2 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0} \qquad \begin{pmatrix} 1-\lambda_3 & -1 & 4 \\ 3 & 2-\lambda_3 & -1 \\ 2 & 1 & -1-\lambda_3 \end{pmatrix} \boldsymbol{\xi}_3 = \mathbf{0}$$

$$\begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0} \qquad \begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0} \qquad \begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix} \boldsymbol{\xi}_3 = \mathbf{0}$$

$$\left(\begin{array}{ccc|c} 0 & -1 & 4 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 3 & -1 & 4 & 0 \\ 0 & 5 & -5 & 0 \\ 5 & 0 & 5 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} -2 & -1 & 4 & 0 \\ 5 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$-\xi_2 + 4\xi_3 = 0 \qquad \xi_1 + \xi_3 = 0 \qquad 5\xi_2 - 5\xi_3 = 0 \qquad 5\xi_1 + 5\xi_3 = 0 \qquad 5\xi_1 - 5\xi_3 = 0 \qquad -2\xi_1 - \xi_2 + 4\xi_3 = 0$$

$$\boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} -\xi_3 \\ 4\xi_3 \\ \xi_3 \end{pmatrix} = \xi_3 \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \qquad \boldsymbol{\xi}_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} -\xi_3 \\ \xi_3 \\ \xi_3 \end{pmatrix} = \xi_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \qquad \boldsymbol{\xi}_3 = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_3 \\ 2\xi_3 \\ \xi_3 \end{pmatrix} = \xi_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Three solutions to the system are then

$$\mathbf{x}_1 = e^{\lambda_1 t} \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{\lambda_2 t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_3 = e^{\lambda_3 t} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Therefore, by the principle of superposition, the general solution is

$$\mathbf{x} = C_1 e^{t} \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{3t} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

where C_1 and C_2 and C_3 are arbitrary constants.