Problem 16

In each of Problems 15 through 18, solve the given initial value problem. Describe the behavior of the solution as $t \to \infty$.

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Solution

Because this is a constant-coefficient homogeneous linear system, it's expected to have solutions of the form $\mathbf{x} = e^{\lambda t}\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ has constant elements.

$$\lambda e^{\lambda t}\boldsymbol{\xi} = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} e^{\lambda t}\boldsymbol{\xi}$$

Divide both sides by $e^{\lambda t}$.

$$\lambda\boldsymbol{\xi} = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \boldsymbol{\xi}$$

This is now an eigenvalue problem.

$$\left\{\begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right\} \boldsymbol{\xi} = \mathbf{0}$$

$$\begin{pmatrix} -2-\lambda & 1 \\ -5 & 4-\lambda \end{pmatrix} \boldsymbol{\xi} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det \begin{pmatrix} -2-\lambda & 1 \\ -5 & 4-\lambda \end{pmatrix} = 0.$$

Evaluate the determinant and solve for λ.

$$(-2-\lambda)(4-\lambda) + 5 = 0$$

$$-3 - 2\lambda + \lambda^2 = 0$$

$$(\lambda+1)(\lambda-3) = 0$$

$$\lambda = \{-1, 3\}$$

Let

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = 3.$$

Substitute these two eigenvalues into equation (1) to determine the corresponding eigenvectors.

$$\begin{pmatrix} -2-\lambda_1 & 1 \\ -5 & 4-\lambda_1 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0} \qquad \begin{pmatrix} -2-\lambda_2 & 1 \\ -5 & 4-\lambda_2 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0}$$

$$\begin{pmatrix} -1 & 1 \\ -5 & 5 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0} \qquad \begin{pmatrix} -5 & 1 \\ -5 & 1 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0}$$

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{cc|c} -5 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$-\xi_1 + \xi_2 = 0 \qquad -5\xi_1 + \xi_2 = 0$$

$$\boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \boldsymbol{\xi}_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ 5\xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

Two solutions to the system are then

$$\mathbf{x}_1 = e^{\lambda_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{\lambda_2 t} \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

By the principle of superposition, the general solution is

$$\mathbf{x} = C_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 5 \end{pmatrix},$$

where C_1 and C_2 are arbitrary constants. Apply the given initial condition to determine them.

$$\mathbf{x}(0) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Write the implied system of equations.

$$\begin{aligned} C_1 + C_2 &= 1 \\ C_1 + 5C_2 &= 3 \end{aligned}$$

Solving it yields $C_1 = 1/2$ and $C_2 = 1/2$. Therefore, the solution to the initial value problem is

$$\mathbf{x} = \frac{1}{2} e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} e^{3t} \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

The first term is the dominant one because e^{3t} increases faster than e^{-t}. That means the solution blows up to $+\infty$ as $t \to \infty$.