Problem 18

In each of Problems 15 through 18, solve the given initial value problem. Describe the behavior of the solution as $t \to \infty$.

$$\mathbf{x}' = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix}$$

Solution

Because this is a constant-coefficient homogeneous linear system, it's expected to have solutions of the form $\mathbf{x} = e^{\lambda t}\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ has constant elements.

$$\lambda e^{\lambda t}\boldsymbol{\xi} = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix} e^{\lambda t}\boldsymbol{\xi}$$

Divide both sides by $e^{\lambda t}$.

$$\lambda\boldsymbol{\xi} = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix} \boldsymbol{\xi}$$

This is now an eigenvalue problem.

$$\left\{ \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right\} \boldsymbol{\xi} = \mathbf{0}$$

$$\begin{pmatrix} -\lambda & 0 & -1 \\ 2 & -\lambda & 0 \\ -1 & 2 & 4-\lambda \end{pmatrix} \boldsymbol{\xi} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det \begin{pmatrix} -\lambda & 0 & -1 \\ 2 & -\lambda & 0 \\ -1 & 2 & 4-\lambda \end{pmatrix} = 0.$$

Evaluate the determinant and solve for λ.

$$(-\lambda) \begin{vmatrix} -\lambda & 0 \\ 2 & 4-\lambda \end{vmatrix} - 1 \begin{vmatrix} 2 & -\lambda \\ -1 & 2 \end{vmatrix} = 0$$

$$(-\lambda)[(-\lambda)(4-\lambda) - 0] - 1[(2)(2) + 1(-\lambda)] = 0$$

$$-4 + \lambda + 4\lambda^2 - \lambda^3 = 0$$

$$(\lambda + 1)(\lambda - 1)(4 - \lambda) = 0$$

$$\lambda = \{-1, 1, 4\}$$

Let

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = -1 \quad \text{and} \quad \lambda_3 = 4.$$

Substitute these three eigenvalues into equation (1) to determine the corresponding eigenvectors.

$$\begin{pmatrix} -\lambda_1 & 0 & -1 \\ 2 & -\lambda_1 & 0 \\ -1 & 2 & 4-\lambda_1 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0} \qquad \begin{pmatrix} -\lambda_2 & 0 & -1 \\ 2 & -\lambda_2 & 0 \\ -1 & 2 & 4-\lambda_2 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0} \qquad \begin{pmatrix} -\lambda_3 & 0 & -1 \\ 2 & -\lambda_3 & 0 \\ -1 & 2 & 4-\lambda_3 \end{pmatrix} \boldsymbol{\xi}_3 = \mathbf{0}$$

$$\begin{pmatrix} -1 & 0 & -1 \\ 2 & -1 & 0 \\ -1 & 2 & 3 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0} \qquad \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 5 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0} \qquad \begin{pmatrix} -4 & 0 & -1 \\ 2 & -4 & 0 \\ -1 & 2 & 0 \end{pmatrix} \boldsymbol{\xi}_3 = \mathbf{0}$$

$$\left(\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 2 & -1 & 0 & 0 \\ -1 & 2 & 3 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 2 & 5 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} -4 & 0 & -1 & 0 \\ 2 & -4 & 0 & 0 \\ -1 & 2 & 0 & 0 \end{array}\right)$$

$$-\xi_1 - \xi_3 = 0 \quad 2\xi_1 - \xi_2 = 0 \qquad \xi_1 - \xi_3 = 0 \quad 2\xi_1 + \xi_2 = 0 \qquad -4\xi_1 - \xi_3 = 0 \quad -\xi_1 + 2\xi_2 = 0$$

$$\boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ 2\xi_1 \\ -\xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \qquad \boldsymbol{\xi}_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ -2\xi_1 \\ \xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad \boldsymbol{\xi}_3 = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \frac{1}{2}\xi_1 \\ -4\xi_1 \end{pmatrix} = \xi_1' \begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix}$$

Three solutions to the system are then

$$\mathbf{x}_1 = e^{\lambda_1 t} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{\lambda_2 t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_3 = e^{\lambda_3 t} \begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix}.$$

By the principle of superposition, the general solution is

$$\mathbf{x} = C_1 e^{t} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + C_3 e^{4t} \begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix},$$

where C_1 and C_2 and C_3 are arbitrary constants.

Apply the given initial condition to determine them.

$$\mathbf{x}(0) = C_1 e^t \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + C_3 e^{4t} \begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix}$$

Write the implied system of equations.

$$\begin{aligned} C_1 + C_2 + 2C_3 &= 7 \\ 2C_1 - 2C_2 + C_3 &= 5 \\ -C_1 + C_2 - 8C_3 &= 5 \end{aligned}$$

Solving it yields $C_1 = 6$ and $C_2 = 3$ and $C_3 = -1$. Therefore, the solution to the initial value problem is

$$\mathbf{x} = 6e^t \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 3e^{-t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} - e^{4t} \begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix}.$$

The third term is the dominant one because e^{4t} increases faster than e^t or e^{-t}. That means the first two components fall down to $-\infty$ and the third component blows up to $+\infty$ as $t \to \infty$.