Problem 19

The system $t\mathbf{x}' = \mathbf{A}\mathbf{x}$ is analogous to the second order Euler equation (Section 5.4). Assuming that $\mathbf{x} = \boldsymbol{\xi} t^r$, where $\boldsymbol{\xi}$ is a constant vector, show that $\boldsymbol{\xi}$ and r must satisfy $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$ in order to obtain nontrivial solutions of the given differential equation.

Solution

Because this looks like an Euler equation, make the substitution

$$\mathbf{x} = \boldsymbol{\xi} t^r,$$

where $\boldsymbol{\xi}$ is a constant.

$$t(\boldsymbol{\xi} r t^{r-1}) = \mathbf{A}\boldsymbol{\xi} t^r$$

$$r t^r \boldsymbol{\xi} = \mathbf{A}\boldsymbol{\xi} t^r$$

Divide both sides by t^r.

$$r\boldsymbol{\xi} = \mathbf{A}\boldsymbol{\xi}$$

Bring both terms to the right side.

$$\begin{aligned} \mathbf{0} &= \mathbf{A}\boldsymbol{\xi} - r\boldsymbol{\xi} \\ &= \mathbf{A}\boldsymbol{\xi} - r\mathbf{I}\boldsymbol{\xi} \\ &= (\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} \end{aligned}$$

The system of ODEs has therefore become an eigenvalue problem.