Problem 2

In each of Problems 1 through 6:

(a) Find the general solution of the given system of equations and describe the behavior of the solution as $t \to \infty$.

(b) Draw a direction field and plot a few trajectories of the system.

\[ x' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} x \]

Solution

Because this is a constant-coefficient homogeneous linear system, it’s expected to have solutions of the form $x = e^{\lambda t} \xi$, where $\xi$ has constant elements.

\[ \lambda e^{\lambda t} \xi = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} e^{\lambda t} \xi \]

Divide both sides by $e^{\lambda t}$.

\[ \lambda \xi = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \xi \]

This is now an eigenvalue problem.

\[ \begin{vmatrix} 1 - \lambda & -2 \\ 3 & -4 - \lambda \end{vmatrix} \xi = 0 \]

The eigenvalues satisfy

\[ \det \begin{pmatrix} 1 - \lambda & -2 \\ 3 & -4 - \lambda \end{pmatrix} = 0. \]

Evaluate the determinant and solve for $\lambda$.

\[ (1 - \lambda)(-4 - \lambda) + 6 = 0 \]
\[ 2 + 3\lambda + \lambda^2 = 0 \]
\[ (\lambda + 2)(\lambda + 1) = 0 \]
\[ \lambda = \{-2, -1\} \]

Let

\[ \lambda_1 = -1 \quad \text{and} \quad \lambda_2 = -2. \]
Substitute these two eigenvalues into equation (1) to determine the corresponding eigenvectors.

\[
\begin{pmatrix}
1 - \lambda_1 & -2 \\
3 & -4 - \lambda_1
\end{pmatrix} \xi_1 = 0
\]
\[
\begin{pmatrix}
1 - \lambda_2 & -2 \\
3 & -4 - \lambda_2
\end{pmatrix} \xi_2 = 0
\]

\[
\begin{pmatrix}
2 \\
3
\end{pmatrix} \xi_1 = 0
\]
\[
\begin{pmatrix}
3 \\
3
\end{pmatrix} \xi_2 = 0
\]

\[
2\xi_1 - 2\xi_2 = 0
\]
\[
3\xi_1 - 2\xi_2 = 0
\]

\[
\xi_1 = \xi_2
\]
\[
\xi_1 = \frac{2}{3} \xi_2
\]

\[
\xi_1 = \left( \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right) = \left( \frac{1}{1} \right) \quad \xi_2 = \left( \frac{2/3 \xi_2}{\xi_2} \right) = \left( \frac{2}{3} \right) = \left( \frac{2}{3} \right)
\]

Two solutions to the system are then

\[
x_1 = e^{\lambda_1 t} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) = \left( e^{-t} \right) \quad \text{and} \quad x_2 = e^{\lambda_2 t} \left( \begin{array}{c} 2 \\ 3 \end{array} \right) = \left( 2e^{-2t} \right).
\]

The Wronskian of these two functions is

\[
W[x_1, x_2](t) = \begin{vmatrix} e^{-t} & 2e^{-2t} \\ e^{-t} & 3e^{-2t} \end{vmatrix} = 3e^{-3t} - 2e^{-3t} = e^{-3t},
\]

which is never zero. That means \(x_1\) and \(x_2\) form a fundamental set of solutions. Therefore, by the principle of superposition, the general solution is

\[
x = C_1 e^{-t} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) + C_2 e^{-2t} \left( \begin{array}{c} 2 \\ 3 \end{array} \right),
\]

where \(C_1\) and \(C_2\) are arbitrary constants. The direction field is the two-dimensional vector field obtained from the system of equations.

\[
x' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} x = \begin{pmatrix} x_1 - 2x_2 \\ 3x_1 - 4x_2 \end{pmatrix} \Rightarrow \langle x_1 - 2x_2, 3x_1 - 4x_2 \rangle
\]
The direction field is shown below in red, and the possible trajectories are shown below in blue.