Problem 2

In each of Problems 1 through 6:

(a) Find the general solution of the given system of equations and describe the behavior of the solution as $t \to \infty$.

(b) Draw a direction field and plot a few trajectories of the system.

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$$

Solution

Because this is a constant-coefficient homogeneous linear system, it's expected to have solutions of the form $\mathbf{x} = e^{\lambda t}\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ has constant elements.

$$\lambda e^{\lambda t}\boldsymbol{\xi} = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} e^{\lambda t}\boldsymbol{\xi}$$

Divide both sides by $e^{\lambda t}$.

$$\lambda\boldsymbol{\xi} = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \boldsymbol{\xi}$$

This is now an eigenvalue problem.

$$\left\{\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right\} \boldsymbol{\xi} = \mathbf{0}$$

$$\begin{pmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{pmatrix} \boldsymbol{\xi} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det \begin{pmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{pmatrix} = 0.$$

Evaluate the determinant and solve for λ.

$$(1-\lambda)(-4-\lambda) + 6 = 0$$

$$2 + 3\lambda + \lambda^2 = 0$$

$$(\lambda+2)(\lambda+1) = 0$$

$$\lambda = \{-2, -1\}$$

Let

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = -2.$$

Substitute these two eigenvalues into equation (1) to determine the corresponding eigenvectors.

$$\begin{pmatrix} 1-\lambda_1 & -2 \\ 3 & -4-\lambda_1 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0} \qquad \begin{pmatrix} 1-\lambda_2 & -2 \\ 3 & -4-\lambda_2 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0}$$

$$\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0} \qquad \begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0}$$

$$2\xi_1 - 2\xi_2 = 0 \qquad 3\xi_1 - 2\xi_2 = 0$$

$$\xi_1 = \xi_2 \qquad \xi_1 = \frac{2}{3}\xi_2$$

$$\boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \boldsymbol{\xi}_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}\xi_2 \\ \xi_2 \end{pmatrix} = \xi_2 \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} = \xi_2' \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Two solutions to the system are then

$$\mathbf{x}_1 = e^{\lambda_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{\lambda_2 t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2e^{-2t} \\ 3e^{-2t} \end{pmatrix}.$$

The Wronskian of these two functions is

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} e^{-t} & 2e^{-2t} \\ e^{-t} & 3e^{-2t} \end{vmatrix} = 3e^{-3t} - 2e^{-3t} = e^{-3t},$$

which is never zero. That means $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental set of solutions. Therefore, by the principle of superposition, the general solution is

$$\mathbf{x} = C_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

where C_1 and C_2 are arbitrary constants. The direction field is the two-dimensional vector field obtained from the system of equations.

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} x_1 - 2x_2 \\ 3x_1 - 4x_2 \end{pmatrix} \quad \Rightarrow \quad \langle x_1 - 2x_2, 3x_1 - 4x_2 \rangle$$

The direction field is shown below in red, and the possible trajectories are shown below in blue.