Problem 20

Referring to Problem 19, solve the given system of equations in each of Problems 20 through 23. Assume that $t > 0$.

$$t\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}$$

Solution

Because this is an Euler linear system, it's expected to have solutions of the form $\mathbf{x} = t^\lambda \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ has constant elements.

$$t(\lambda t^{\lambda-1}\boldsymbol{\xi}) = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} t^\lambda \boldsymbol{\xi}$$

Divide both sides by t^λ.

$$\lambda\boldsymbol{\xi} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \boldsymbol{\xi}$$

This is now an eigenvalue problem.

$$\left\{\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right\} \boldsymbol{\xi} = \mathbf{0}$$

$$\begin{pmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{pmatrix} \boldsymbol{\xi} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det\begin{pmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{pmatrix} = 0.$$

Evaluate the determinant and solve for λ.

$$(2-\lambda)(-2-\lambda) + 3 = 0$$

$$\lambda^2 - 1 = 0$$

$$(\lambda+1)(\lambda-1) = 0$$

$$\lambda = \{-1, 1\}$$

Let

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = 1.$$

Substitute these two eigenvalues into equation (1) to determine the corresponding eigenvectors.

$$\begin{pmatrix} 2-\lambda_1 & -1 \\ 3 & -2-\lambda_1 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0} \qquad \begin{pmatrix} 2-\lambda_2 & -1 \\ 3 & -2-\lambda_2 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0}$$

$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0} \qquad \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0}$$

$$\left(\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$3\xi_1 - \xi_2 = 0 \qquad \xi_1 - \xi_2 = 0$$

$$\boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ 3\xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \boldsymbol{\xi}_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Two solutions to the system are then

$$\mathbf{x}_1 = t^{\lambda_1} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = t^{\lambda_2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, by the principle of superposition, the general solution is

$$\mathbf{x} = C_1 t^{-1} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 t^1 \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where C_1 and C_2 are arbitrary constants.