

## Problem 21

Referring to Problem 19, solve the given system of equations in each of Problems 20 through 23. Assume that  $t > 0$ .

$$t\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}$$

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### Solution

Because this is an Euler linear system, it's expected to have solutions of the form  $\mathbf{x} = t^\lambda \boldsymbol{\xi}$ , where  $\boldsymbol{\xi}$  has constant elements.

$$t(\lambda t^{\lambda-1} \boldsymbol{\xi}) = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} t^\lambda \boldsymbol{\xi}$$

Divide both sides by  $t^\lambda$ .

$$\lambda \boldsymbol{\xi} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \boldsymbol{\xi}$$

This is now an eigenvalue problem.

$$\left\{ \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right\} \boldsymbol{\xi} = \mathbf{0}$$
$$\begin{pmatrix} 5 - \lambda & -1 \\ 3 & 1 - \lambda \end{pmatrix} \boldsymbol{\xi} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det \begin{pmatrix} 5 - \lambda & -1 \\ 3 & 1 - \lambda \end{pmatrix} = 0.$$

Evaluate the determinant and solve for  $\lambda$ .

$$(5 - \lambda)(1 - \lambda) + 3 = 0$$

$$8 - 6\lambda + \lambda^2 = 0$$

$$(\lambda - 2)(\lambda - 4) = 0$$

$$\lambda = \{2, 4\}$$

Let

$$\lambda_1 = 4 \quad \text{and} \quad \lambda_2 = 2.$$

Substitute these two eigenvalues into equation (1) to determine the corresponding eigenvectors.

$$\begin{aligned} \begin{pmatrix} 5 - \lambda_1 & -1 \\ 3 & 1 - \lambda_1 \end{pmatrix} \boldsymbol{\xi}_1 &= \mathbf{0} & \begin{pmatrix} 5 - \lambda_2 & -1 \\ 3 & 1 - \lambda_2 \end{pmatrix} \boldsymbol{\xi}_2 &= \mathbf{0} \\ \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \boldsymbol{\xi}_1 &= \mathbf{0} & \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \boldsymbol{\xi}_2 &= \mathbf{0} \\ \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) & & \left( \begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) & \\ \xi_1 - \xi_2 &= 0 & 3\xi_1 - \xi_2 &= 0 \\ \boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} & & \boldsymbol{\xi}_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ 3\xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} \end{aligned}$$

Two solutions to the system are then

$$\mathbf{x}_1 = t^{\lambda_1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = t^{\lambda_2} \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Therefore, by the principle of superposition, the general solution is

$$\mathbf{x} = C_1 t^4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 t^2 \begin{pmatrix} 1 \\ 3 \end{pmatrix},$$

where  $C_1$  and  $C_2$  are arbitrary constants.