

Problem 23

Referring to Problem 19, solve the given system of equations in each of Problems 20 through 23. Assume that $t > 0$.

$$t\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$$

Solution

Because this is an Euler linear system, it's expected to have solutions of the form $\mathbf{x} = t^\lambda \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ has constant elements.

$$t(\lambda t^{\lambda-1} \boldsymbol{\xi}) = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} t^\lambda \boldsymbol{\xi}$$

Divide both sides by t^λ .

$$\lambda \boldsymbol{\xi} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \boldsymbol{\xi}$$

This is now an eigenvalue problem.

$$\begin{aligned} \left\{ \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right\} \boldsymbol{\xi} &= \mathbf{0} \\ \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} \boldsymbol{\xi} &= \mathbf{0} \end{aligned} \tag{1}$$

The eigenvalues satisfy

$$\det \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} = 0.$$

Evaluate the determinant and solve for λ .

$$(3-\lambda)(-2-\lambda) + 4 = 0$$

$$-2 - \lambda + \lambda^2 = 0$$

$$(\lambda + 1)(\lambda - 2) = 0$$

$$\lambda = \{-1, 2\}$$

Let

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = 2.$$

Substitute these two eigenvalues into equation (1) to determine the corresponding eigenvectors.

$$\begin{aligned} \begin{pmatrix} 3 - \lambda_1 & -2 \\ 2 & -2 - \lambda_1 \end{pmatrix} \boldsymbol{\xi}_1 &= \mathbf{0} & \begin{pmatrix} 3 - \lambda_2 & -2 \\ 2 & -2 - \lambda_2 \end{pmatrix} \boldsymbol{\xi}_2 &= \mathbf{0} \\ \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \boldsymbol{\xi}_1 &= \mathbf{0} & \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \boldsymbol{\xi}_2 &= \mathbf{0} \\ \left(\begin{array}{cc|c} 4 & -2 & 0 \\ 2 & -1 & 0 \end{array} \right) & & \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & -4 & 0 \end{array} \right) & \\ 4\xi_1 - 2\xi_2 = 0 & & \xi_1 - 2\xi_2 = 0 & \\ \boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ 2\xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} & & \boldsymbol{\xi}_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2\xi_2 \\ \xi_2 \end{pmatrix} = \xi_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} & \end{aligned}$$

Two solutions to the system are then

$$\mathbf{x}_1 = t^{\lambda_1} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = t^{\lambda_2} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Therefore, by the principle of superposition, the general solution is

$$\mathbf{x} = C_1 t^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 t^2 \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

where C_1 and C_2 are arbitrary constants.