Problem 24

In each of Problems 24 through 27, the eigenvalues and eigenvectors of a matrix $\mathbf{A}$ are given. Consider the corresponding system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(a) Sketch a phase portrait of the system.

(b) Sketch the trajectory passing through the initial point $(2, 3)$.

(c) For the trajectory in part (b), sketch the graphs of x_1 versus t and of x_2 versus t on the same set of axes.

$$r_1 = -1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \qquad r_2 = -2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solution

Because this is a constant-coefficient homogeneous linear system, it's expected to have solutions of the form $\mathbf{x} = e^{\lambda t}\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ has constant elements.

$$\lambda e^{\lambda t}\boldsymbol{\xi} = \mathbf{A}e^{\lambda t}\boldsymbol{\xi}$$

Divide both sides by $e^{\lambda t}$.

$$\lambda\boldsymbol{\xi} = \mathbf{A}\boldsymbol{\xi}$$

This is now an eigenvalue problem.

$$(\mathbf{A} - \lambda\mathbf{I})\xi = \mathbf{0}$$

$$\begin{pmatrix} A_{11} - \lambda & A_{12} \\ A_{21} & A_{22} - \lambda \end{pmatrix} \xi = \mathbf{0}$$

The elements of $\mathbf{A}$ can be deduced from the provided eigenvalues and eigenvectors.

$$\begin{pmatrix} A_{11} - \lambda_1 & A_{12} \\ A_{21} & A_{22} - \lambda_1 \end{pmatrix} \xi_1 = \mathbf{0} \qquad \begin{pmatrix} A_{11} - \lambda_2 & A_{12} \\ A_{21} & A_{22} - \lambda_2 \end{pmatrix} \xi_2 = \mathbf{0}$$

$$\begin{pmatrix} A_{11} + 1 & A_{12} \\ A_{21} & A_{22} + 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \mathbf{0} \qquad \begin{pmatrix} A_{11} + 2 & A_{12} \\ A_{21} & A_{22} + 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \mathbf{0}$$

$$\left.\begin{aligned} -A_{11} - 1 + 2A_{12} &= 0 \\ -A_{21} + 2A_{22} + 2 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} A_{11} + 2 + 2A_{12} &= 0 \\ A_{21} + 2A_{22} + 4 &= 0 \end{aligned}\right\}$$

Solve these four equations for the matrix elements.

$$A_{11} = -\frac{3}{2} \quad \text{and} \quad A_{12} = -\frac{1}{4} \quad \text{and} \quad A_{21} = -1 \quad \text{and} \quad A_{22} = -\frac{3}{2}$$

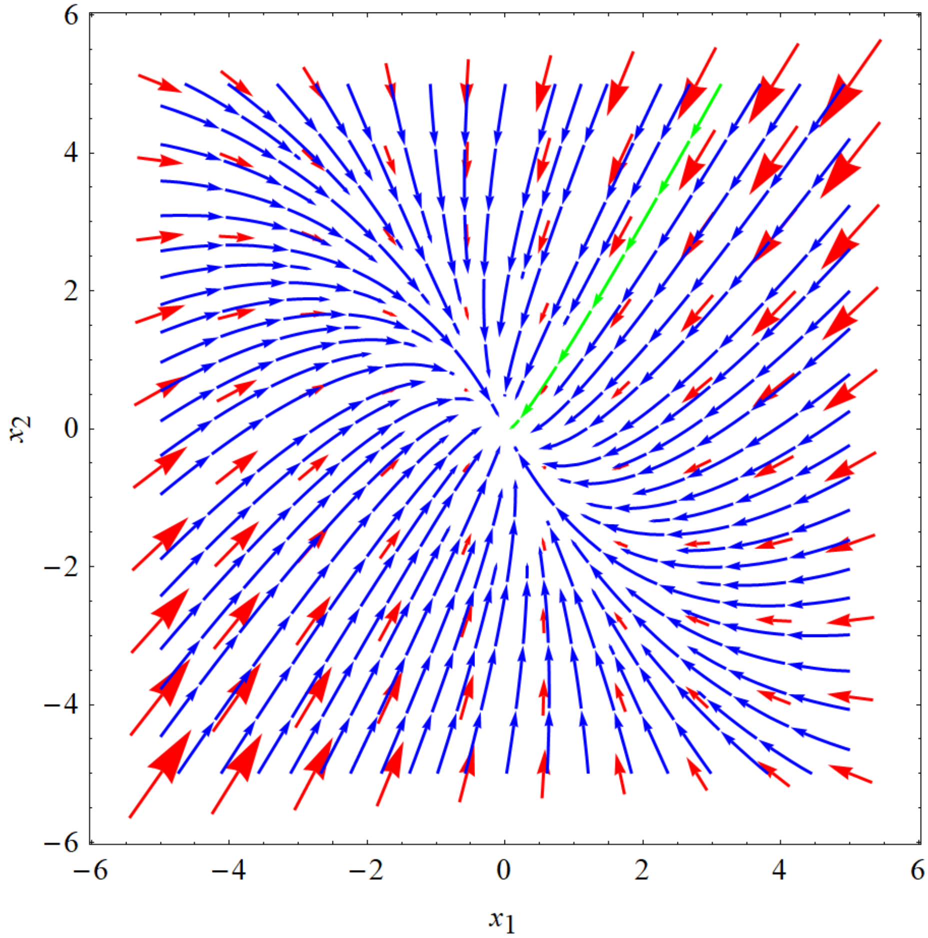
As a result, the system of ODEs is

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & -\frac{1}{4} \\ -1 & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}x_1 - \frac{1}{4}x_2 \\ -x_1 - \frac{3}{2}x_2 \end{pmatrix}.$$

The direction field, the red two-dimensional vector field

$$\left\langle -\frac{3}{2}x_1 - \frac{1}{4}x_2, -x_1 - \frac{3}{2}x_2 \right\rangle,$$

and phase portrait (blue streamlines) of the system can now be drawn.



The trajectory passing through $(2, 3)$ is shown in green. Use the principle of superposition to get the general solution to the system of ODEs.

$$\begin{aligned}\mathbf{x} &= C_1 e^{\lambda_1 t}\xi_1 + C_2 e^{\lambda_2 t}\xi_2 \\ &= C_1 e^{-t}\begin{pmatrix}-1\\2\end{pmatrix} + C_2 e^{-2t}\begin{pmatrix}1\\2\end{pmatrix}\end{aligned}$$

Determine C_1 and C_2 for this special trajectory.

$$\mathbf{x}(0) = C_1\begin{pmatrix}-1\\2\end{pmatrix} + C_2\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$$

Write the implied system of equations.

$$\begin{aligned}-C_1 + C_2 &= 2 \\ 2C_1 + 2C_2 &= 3\end{aligned}$$

Solve for C_1 and C_2.

$$C_1 = -\frac{1}{4} \quad \text{and} \quad C_2 = \frac{7}{4}$$

Consequently, the solution is

$$\mathbf{x} = -\frac{1}{4}e^{-t}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \frac{7}{4}e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}e^{-t} + \frac{7}{4}e^{-2t} \\ -\frac{1}{2}e^{-t} + \frac{7}{2}e^{-2t} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The graph of x_1 and x_2 versus t is shown below.