Problem 26

In each of Problems 24 through 27, the eigenvalues and eigenvectors of a matrix \( A \) are given. Consider the corresponding system \( x' = Ax \).

(a) Sketch a phase portrait of the system.

(b) Sketch the trajectory passing through the initial point \((2, 3)\).

(c) For the trajectory in part (b), sketch the graphs of \( x_1 \) versus \( t \) and of \( x_2 \) versus \( t \) on the same set of axes.

\[ r_1 = -1, \quad \xi^{(1)} = \left( \begin{array}{c} -1 \\ 2 \end{array} \right); \quad r_2 = 2, \quad \xi^{(2)} = \left( \begin{array}{c} 1 \\ 2 \end{array} \right) \]

Solution

Because this is a constant-coefficient homogeneous linear system, it’s expected to have solutions of the form \( x = e^{\lambda t}\xi \), where \( \xi \) has constant elements.

\[ \lambda e^{\lambda t}\xi = Ae^{\lambda t}\xi \]

Divide both sides by \( e^{\lambda t} \).

\[ \lambda \xi = A\xi \]

This is now an eigenvalue problem.

\[ (A - \lambda I)\xi = 0 \]

\[ \left( \begin{array}{cc} A_{11} - \lambda & A_{12} \\ A_{21} & A_{22} - \lambda \end{array} \right) \xi = 0 \]

The elements of \( A \) can be deduced from the provided eigenvalues and eigenvectors.

\[ \left( \begin{array}{cc} A_{11} - \lambda_1 & A_{12} \\ A_{21} & A_{22} - \lambda_1 \end{array} \right) \xi_1 = 0 \quad \left( \begin{array}{cc} A_{11} - \lambda_2 & A_{12} \\ A_{21} & A_{22} - \lambda_2 \end{array} \right) \xi_2 = 0 \]

\[ \left( \begin{array}{cc} A_{11} + 1 & A_{12} \\ A_{21} & A_{22} + 1 \end{array} \right) \left( \begin{array}{c} -1 \\ 2 \end{array} \right) = 0 \quad \left( \begin{array}{cc} A_{11} - 2 & A_{12} \\ A_{21} & A_{22} - 2 \end{array} \right) \left( \begin{array}{c} 1 \\ 2 \end{array} \right) = 0 \]

\[ \begin{align*}
-A_{11} - 1 + 2A_{12} &= 0 \\
-A_{21} + 2A_{22} + 2 &= 0
\end{align*} \quad \begin{align*}
A_{11} - 2 + 2A_{12} &= 0 \\
A_{21} + 2A_{22} - 4 &= 0
\end{align*} \]

Solve these four equations for the matrix elements.

\[ A_{11} = \frac{1}{2} \quad \text{and} \quad A_{12} = \frac{3}{4} \quad \text{and} \quad A_{21} = 3 \quad \text{and} \quad A_{22} = \frac{1}{2} \]

As a result, the system of ODEs is

\[ \left( \begin{array}{c} x_1' \\ x_2' \end{array} \right) = \left( \begin{array}{cc} \frac{1}{2} & \frac{3}{4} \\ 3 & \frac{1}{2} \end{array} \right) \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left( \begin{array}{c} \frac{1}{2}x_1 + \frac{3}{4}x_2 \\ 3x_1 + \frac{1}{2}x_2 \end{array} \right) \]
The direction field, the red two-dimensional vector field
\[
\left( \frac{1}{2} x_1 + \frac{3}{4} x_2, 3 x_1 + \frac{1}{2} x_2 \right),
\]
and phase portrait (blue streamlines) of the system can now be drawn.

The trajectory passing through (2, 3) is shown in green. Use the principle of superposition to get the general solution to the system of ODEs.

\[
x = C_1 e^{\lambda_1 t} \xi_1 + C_2 e^{\lambda_2 t} \xi_2
\]
\[
= C_1 e^{-t} \left( \frac{-1}{2} \right) + C_2 e^{2t} \left( \frac{1}{2} \right)
\]

Determine \( C_1 \) and \( C_2 \) for this special trajectory.

\[
x(0) = C_1 \left( \frac{-1}{2} \right) + C_2 \left( \frac{1}{2} \right) = \left( \frac{2}{3} \right)
\]

Write the implied system of equations.

\[
-C_1 + C_2 = 2
\]
\[
2C_1 + 2C_2 = 3
\]
Solve for $C_1$ and $C_2$.

$$C_1 = -\frac{1}{4} \quad \text{and} \quad C_2 = \frac{7}{4}$$

Consequently, the solution is

$$x = -\frac{1}{4} e^{-t} \left( -\frac{1}{2} \right) + \frac{7}{4} e^{2t} \left( \frac{1}{2} \right) = \left( \frac{1}{4} e^{-t} + \frac{7}{4} e^{2t} \right) = \left( x_1 \right).$$

The graph of $x_1$ and $x_2$ versus $t$ is shown below.