Problem 3

In each of Problems 1 through 6:

(a) Find the general solution of the given system of equations and describe the behavior of the solution as $t \to \infty$.

(b) Draw a direction field and plot a few trajectories of the system.

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}$$

Solution

Because this is a constant-coefficient homogeneous linear system, it's expected to have solutions of the form $\mathbf{x} = e^{\lambda t}\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ has constant elements.

$$\lambda e^{\lambda t}\boldsymbol{\xi} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} e^{\lambda t}\boldsymbol{\xi}$$

Divide both sides by $e^{\lambda t}$.

$$\lambda\boldsymbol{\xi} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \boldsymbol{\xi}$$

This is now an eigenvalue problem.

$$\left\{\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right\}\boldsymbol{\xi} = \mathbf{0}$$

$$\begin{pmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{pmatrix}\boldsymbol{\xi} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det\begin{pmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{pmatrix} = 0.$$

Evaluate the determinant and solve for λ.

$$(2-\lambda)(-2-\lambda) + 3 = 0$$
$$\lambda^2 - 1 = 0$$
$$(\lambda+1)(\lambda-1) = 0$$
$$\lambda = \{-1, 1\}$$

Let

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = -1.$$

Substitute these two eigenvalues into equation (1) to determine the corresponding eigenvectors.

$$\begin{pmatrix} 2-\lambda_1 & -1 \\ 3 & -2-\lambda_1 \end{pmatrix}\boldsymbol{\xi}_1 = \mathbf{0} \qquad \begin{pmatrix} 2-\lambda_2 & -1 \\ 3 & -2-\lambda_2 \end{pmatrix}\boldsymbol{\xi}_2 = \mathbf{0}$$

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\boldsymbol{\xi}_1 = \mathbf{0} \qquad \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\boldsymbol{\xi}_2 = \mathbf{0}$$

$$\xi_1 - \xi_2 = 0 \qquad 3\xi_1 - \xi_2 = 0$$

$$\xi_1 = \xi_2 \qquad \xi_2 = 3\xi_1$$

$$\boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \boldsymbol{\xi}_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ 3\xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Two solutions to the system are then

$$\mathbf{x}_1 = e^{\lambda_1 t}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}e^t\\e^t\end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{\lambda_2 t}\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}e^{-t}\\3e^{-t}\end{pmatrix}.$$

The Wronskian of these two functions is

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix}e^t & e^{-t}\\ e^t & 3e^{-t}\end{vmatrix} = 3 - 1 = 2,$$

which is never zero. That means $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental set of solutions. Therefore, by the principle of superposition, the general solution is

$$\mathbf{x} = C_1 e^t \begin{pmatrix}1\\1\end{pmatrix} + C_2 e^{-t}\begin{pmatrix}1\\3\end{pmatrix},$$

where C_1 and C_2 are arbitrary constants. The direction field is the two-dimensional vector field obtained from the system of equations.

$$\mathbf{x}' = \begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}\mathbf{x} = \begin{pmatrix}2x_1 - x_2\\3x_1 - 2x_2\end{pmatrix} \quad \Rightarrow \quad \langle 2x_1 - x_2, 3x_1 - 2x_2\rangle$$

The direction field is shown below in red, and the possible trajectories are shown below in blue.