Problem 4

In each of Problems 1 through 6:

(a) Find the general solution of the given system of equations and describe the behavior of the solution as $t \to \infty$.

(b) Draw a direction field and plot a few trajectories of the system.

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$$

Solution

Because this is a constant-coefficient homogeneous linear system, it's expected to have solutions of the form $\mathbf{x} = e^{\lambda t}\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ has constant elements.

$$\lambda e^{\lambda t}\boldsymbol{\xi} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} e^{\lambda t}\boldsymbol{\xi}$$

Divide both sides by $e^{\lambda t}$.

$$\lambda\boldsymbol{\xi} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \boldsymbol{\xi}$$

This is now an eigenvalue problem.

$$\left\{\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right\}\boldsymbol{\xi} = \mathbf{0}$$

$$\begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix}\boldsymbol{\xi} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det\begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix} = 0.$$

Evaluate the determinant and solve for λ.

$$\begin{aligned} (1-\lambda)(-2-\lambda) - 4 &= 0 \\ -6 + \lambda + \lambda^2 &= 0 \\ (\lambda+3)(\lambda-2) &= 0 \\ \lambda &= \{-3, 2\} \end{aligned}$$

Let

$$\lambda_1 = -3 \quad \text{and} \quad \lambda_2 = 2.$$

Substitute these two eigenvalues into equation (1) to determine the corresponding eigenvectors.

$$\begin{aligned} \begin{pmatrix} 1-\lambda_1 & 1 \\ 4 & -2-\lambda_1 \end{pmatrix}\boldsymbol{\xi}_1 &= \mathbf{0} \\ \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\boldsymbol{\xi}_1 &= \mathbf{0} \\ 4\xi_1 + \xi_2 &= 0 \\ \xi_2 &= -4\xi_1 \end{aligned} \qquad \begin{aligned} \begin{pmatrix} 1-\lambda_2 & 1 \\ 4 & -2-\lambda_2 \end{pmatrix}\boldsymbol{\xi}_2 &= \mathbf{0} \\ \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\boldsymbol{\xi}_2 &= \mathbf{0} \\ -\xi_1 + \xi_2 &= 0 \\ \xi_2 &= \xi_1 \end{aligned}$$

$$\boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ -4\xi_1 \end{pmatrix} = \xi_1\begin{pmatrix} 1 \\ -4 \end{pmatrix} \qquad \boldsymbol{\xi}_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \xi_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Two solutions to the system are then

$$\mathbf{x}_1 = e^{\lambda_1 t}\begin{pmatrix}1\\-4\end{pmatrix} = \begin{pmatrix}e^{-3t}\\-4e^{-3t}\end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{\lambda_2 t}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}e^{2t}\\e^{2t}\end{pmatrix}.$$

The Wronskian of these two functions is

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix}e^{-3t} & e^{2t}\\-4e^{-3t} & e^{2t}\end{vmatrix} = e^{-t} + 4e^{-t} = 5e^{-t},$$

which is never zero. That means $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental set of solutions. Therefore, by the principle of superposition, the general solution is

$$\mathbf{x} = C_1 e^{-3t}\begin{pmatrix}1\\-4\end{pmatrix} + C_2 e^{2t}\begin{pmatrix}1\\1\end{pmatrix},$$

where C_1 and C_2 are arbitrary constants. The direction field is the two-dimensional vector field obtained from the system of equations.

$$\mathbf{x}' = \begin{pmatrix}1 & 1\\4 & -2\end{pmatrix}\mathbf{x} = \begin{pmatrix}x_1 + x_2\\4x_1 - 2x_2\end{pmatrix} \quad \Rightarrow \quad \langle x_1 + x_2, 4x_1 - 2x_2\rangle$$

The direction field is shown below in red, and the possible trajectories are shown below in blue.