Problem 4

In each of Problems 1 through 6:

(a) Find the general solution of the given system of equations and describe the behavior of the solution as \( t \to \infty \).

(b) Draw a direction field and plot a few trajectories of the system.

\[
x' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} x
\]

Solution

Because this is a constant-coefficient homogeneous linear system, it’s expected to have solutions of the form \( x = e^{\lambda t} \xi \), where \( \xi \) has constant elements.

\[
\lambda e^{\lambda t} \xi = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} e^{\lambda t} \xi
\]

Divide both sides by \( e^{\lambda t} \).

\[
\lambda \xi = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \xi
\]

This is now an eigenvalue problem.

\[
\left\{ \left( \begin{array}{cc} 1 & 1 \\ 4 & -2 \end{array} \right) - \left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda \end{array} \right) \right\} \xi = 0
\]

\[
\left( \begin{array}{cc} 1 - \lambda & 1 \\ 4 & -2 - \lambda \end{array} \right) \xi = 0
\]

(1)

The eigenvalues satisfy

\[
\det \left( \begin{array}{cc} 1 - \lambda & 1 \\ 4 & -2 - \lambda \end{array} \right) = 0.
\]

Evaluate the determinant and solve for \( \lambda \).

\[
(1 - \lambda)(-2 - \lambda) - 4 = 0
\]

\[-6 + \lambda + \lambda^2 = 0
\]

\[(\lambda + 3)(\lambda - 2) = 0
\]

\[\lambda = \{-3, 2\}
\]

Let \( \lambda_1 = -3 \) and \( \lambda_2 = 2 \).

Substitute these two eigenvalues into equation (1) to determine the corresponding eigenvectors.

\[
\begin{pmatrix} 1 - \lambda_1 & 1 \\ 4 & -2 - \lambda_1 \end{pmatrix} \xi_1 = 0
\]

\[
\begin{pmatrix} 1 - \lambda_2 & 1 \\ 4 & -2 - \lambda_2 \end{pmatrix} \xi_2 = 0
\]

\[
\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \xi_1 = 0
\]

\[
4\xi_1 + \xi_2 = 0
\]

\[
\xi_2 = -4\xi_1
\]

\[
\xi_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix}
\]

\[
\xi_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

www.stemjock.com
Two solutions to the system are then

\[ \mathbf{x}_1 = e^{\lambda_1 t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} e^{-3t} \\ -4e^{-3t} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{\lambda_2 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} . \]

The Wronskian of these two functions is

\[ W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{vmatrix} = e^{-t} + 4e^{-t} = 5e^{-t}, \]

which is never zero. That means \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) form a fundamental set of solutions. Therefore, by the principle of superposition, the general solution is

\[ \mathbf{x} = C_1 e^{-3t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \]

where \( C_1 \) and \( C_2 \) are arbitrary constants. The direction field is the two-dimensional vector field obtained from the system of equations.

\[ \mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} x_1 + x_2 \\ 4x_1 - 2x_2 \end{pmatrix} \Rightarrow \langle x_1 + x_2, 4x_1 - 2x_2 \rangle \]

The direction field is shown below in red, and the possible trajectories are shown below in blue.