

## Problem 5

In each of Problems 1 through 6:

- Find the general solution of the given system of equations and describe the behavior of the solution as  $t \rightarrow \infty$ .
- Draw a direction field and plot a few trajectories of the system.

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x}$$

### Solution

Because this is a constant-coefficient homogeneous linear system, it's expected to have solutions of the form  $\mathbf{x} = e^{\lambda t} \boldsymbol{\xi}$ , where  $\boldsymbol{\xi}$  has constant elements.

$$\lambda e^{\lambda t} \boldsymbol{\xi} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} e^{\lambda t} \boldsymbol{\xi}$$

Divide both sides by  $e^{\lambda t}$ .

$$\lambda \boldsymbol{\xi} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \boldsymbol{\xi}$$

This is now an eigenvalue problem.

$$\left\{ \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right\} \boldsymbol{\xi} = \mathbf{0}$$

$$\begin{pmatrix} -2 - \lambda & 1 \\ 1 & -2 - \lambda \end{pmatrix} \boldsymbol{\xi} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det \begin{pmatrix} -2 - \lambda & 1 \\ 1 & -2 - \lambda \end{pmatrix} = 0.$$

Evaluate the determinant and solve for  $\lambda$ .

$$\begin{aligned} (-2 - \lambda)(-2 - \lambda) - 1 &= 0 \\ 3 + 4\lambda + \lambda^2 &= 0 \\ (\lambda + 3)(\lambda + 1) &= 0 \\ \lambda &= \{-3, -1\} \end{aligned}$$

Let

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = -3.$$

Substitute these two eigenvalues into equation (1) to determine the corresponding eigenvectors.

$$\begin{aligned} \begin{pmatrix} -2 - \lambda_1 & 1 \\ 1 & -2 - \lambda_1 \end{pmatrix} \boldsymbol{\xi}_1 &= \mathbf{0} & \begin{pmatrix} -2 - \lambda_2 & 1 \\ 1 & -2 - \lambda_2 \end{pmatrix} \boldsymbol{\xi}_2 &= \mathbf{0} \\ \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \boldsymbol{\xi}_1 &= \mathbf{0} & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \boldsymbol{\xi}_2 &= \mathbf{0} \\ -\xi_1 + \xi_2 &= 0 & \xi_1 + \xi_2 &= 0 \\ \xi_2 &= \xi_1 & \xi_2 &= -\xi_1 \\ \boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} & & \boldsymbol{\xi}_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ -\xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

Two solutions to the system are then

$$\mathbf{x}_1 = e^{\lambda_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{\lambda_2 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{-3t} \\ -e^{-3t} \end{pmatrix}.$$

The Wronskian of these two functions is

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{vmatrix} = -e^{-4t} - e^{-4t} = -2e^{-4t},$$

which is never zero. That means  $\mathbf{x}_1$  and  $\mathbf{x}_2$  form a fundamental set of solutions. Therefore, by the principle of superposition, the general solution is

$$\mathbf{x} = C_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

where  $C_1$  and  $C_2$  are arbitrary constants. The direction field is the two-dimensional vector field obtained from the system of equations.

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -2x_1 + x_2 \\ x_1 - 2x_2 \end{pmatrix} \Rightarrow \langle -2x_1 + x_2, x_1 - 2x_2 \rangle$$

The direction field is shown below in red, and the possible trajectories are shown below in blue.

