

Problem 6

In each of Problems 1 through 6:

- Find the general solution of the given system of equations and describe the behavior of the solution as $t \rightarrow \infty$.
- Draw a direction field and plot a few trajectories of the system.

$$\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} \mathbf{x}$$

Solution

Because this is a constant-coefficient homogeneous linear system, it's expected to have solutions of the form $\mathbf{x} = e^{\lambda t} \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ has constant elements.

$$\lambda e^{\lambda t} \boldsymbol{\xi} = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} e^{\lambda t} \boldsymbol{\xi}$$

Divide both sides by $e^{\lambda t}$.

$$\lambda \boldsymbol{\xi} = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} \boldsymbol{\xi}$$

This is now an eigenvalue problem.

$$\left\{ \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right\} \boldsymbol{\xi} = \mathbf{0}$$

$$\begin{pmatrix} \frac{5}{4} - \lambda & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} - \lambda \end{pmatrix} \boldsymbol{\xi} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det \begin{pmatrix} \frac{5}{4} - \lambda & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} - \lambda \end{pmatrix} = 0.$$

Evaluate the determinant and solve for λ .

$$(5/4 - \lambda)(5/4 - \lambda) - \frac{9}{16} = 0$$

$$1 - \frac{5}{2}\lambda + \lambda^2 = 0$$

$$\left(\lambda - \frac{1}{2}\right)(\lambda - 2) = 0$$

$$\lambda = \left\{ \frac{1}{2}, 2 \right\}$$

Let

$$\lambda_1 = \frac{1}{2} \quad \text{and} \quad \lambda_2 = 2.$$

Substitute these two eigenvalues into equation (1) to determine the corresponding eigenvectors.

$$\begin{aligned} \begin{pmatrix} \frac{5}{4} - \lambda_1 & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} - \lambda_1 \end{pmatrix} \boldsymbol{\xi}_1 &= \mathbf{0} & \begin{pmatrix} \frac{5}{4} - \lambda_2 & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} - \lambda_2 \end{pmatrix} \boldsymbol{\xi}_2 &= \mathbf{0} \\ \begin{pmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} \end{pmatrix} \boldsymbol{\xi}_1 &= \mathbf{0} & \begin{pmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{3}{4} \end{pmatrix} \boldsymbol{\xi}_2 &= \mathbf{0} \\ \frac{3}{4}\xi_1 + \frac{3}{4}\xi_2 &= 0 & -\frac{3}{4}\xi_1 + \frac{3}{4}\xi_2 &= 0 \\ \xi_2 &= -\xi_1 & \xi_2 &= \xi_1 \\ \boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ -\xi_1 \end{pmatrix} &= \xi_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} & \boldsymbol{\xi}_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} &= \xi_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

Two solutions to the system are then

$$\mathbf{x}_1 = e^{\lambda_1 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{t/2} \\ -e^{t/2} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{\lambda_2 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}.$$

The Wronskian of these two functions is

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} e^{t/2} & e^{2t} \\ -e^{t/2} & e^{2t} \end{vmatrix} = e^{5t/2} + e^{5t/2} = 2e^{5t/2},$$

which is never zero. That means \mathbf{x}_1 and \mathbf{x}_2 form a fundamental set of solutions. Therefore, by the principle of superposition, the general solution is

$$\mathbf{x} = C_1 e^{t/2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where C_1 and C_2 are arbitrary constants. The direction field is the two-dimensional vector field obtained from the system of equations.

$$\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{5}{4}x_1 + \frac{3}{4}x_2 \\ \frac{3}{4}x_1 + \frac{5}{4}x_2 \end{pmatrix} \Rightarrow \left\langle \frac{5}{4}x_1 + \frac{3}{4}x_2, \frac{3}{4}x_1 + \frac{5}{4}x_2 \right\rangle$$

The direction field is shown below in red, and the possible trajectories are shown below in blue.

