Problem 7

In each of Problems 7 and 8:

(a) Find the general solution of the given system of equations.

(b) Draw a direction field and a few of the trajectories. In each of these problems, the coefficient matrix has a zero eigenvalue. As a result, the pattern of trajectories is different from those in the examples in the text.

$$\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \mathbf{x}$$

Solution

Because this is a constant-coefficient homogeneous linear system, it's expected to have solutions of the form $\mathbf{x} = e^{\lambda t}\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ has constant elements.

$$\lambda e^{\lambda t}\boldsymbol{\xi} = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} e^{\lambda t}\boldsymbol{\xi}$$

Divide both sides by $e^{\lambda t}$.

$$\lambda\boldsymbol{\xi} = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \boldsymbol{\xi}$$

This is now an eigenvalue problem.

$$\left\{\begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right\}\boldsymbol{\xi} = \mathbf{0}$$

$$\begin{pmatrix} 4-\lambda & -3 \\ 8 & -6-\lambda \end{pmatrix}\boldsymbol{\xi} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det\begin{pmatrix} 4-\lambda & -3 \\ 8 & -6-\lambda \end{pmatrix} = 0.$$

Evaluate the determinant and solve for λ.

$$(4-\lambda)(-6-\lambda) + 24 = 0$$

$$2\lambda + \lambda^2 = 0$$

$$\lambda(\lambda + 2) = 0$$

$$\lambda = \{-2, 0\}$$

Let

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = -2.$$

Substitute these two eigenvalues into equation (1) to determine the corresponding eigenvectors.

$$\begin{aligned} \begin{pmatrix} 4-\lambda_1 & -3 \\ 8 & -6-\lambda_1 \end{pmatrix} \boldsymbol{\xi}_1 &= \mathbf{0} \\ \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \boldsymbol{\xi}_1 &= \mathbf{0} \\ 4\xi_1 - 3\xi_2 &= 0 \\ \xi_2 &= \frac{4}{3}\xi_1 \\ \boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \frac{4}{3}\xi_1 \end{pmatrix} &= \xi_1' \begin{pmatrix} 3 \\ 4 \end{pmatrix} \end{aligned} \qquad \begin{aligned} \begin{pmatrix} 4-\lambda_2 & -3 \\ 8 & -6-\lambda_2 \end{pmatrix} \boldsymbol{\xi}_2 &= \mathbf{0} \\ \begin{pmatrix} 6 & -3 \\ 8 & -4 \end{pmatrix} \boldsymbol{\xi}_2 &= \mathbf{0} \\ 6\xi_1 - 3\xi_2 &= 0 \\ \xi_2 &= 2\xi_1 \\ \boldsymbol{\xi}_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ 2\xi_1 \end{pmatrix} &= \xi_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

Two solutions to the system are then

$$\mathbf{x}_1 = e^{\lambda_1 t} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{\lambda_2 t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix}.$$

The Wronskian of these two functions is

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} 3 & e^{-2t} \\ 4 & 2e^{-2t} \end{vmatrix} = 6e^{-2t} - 4e^{-2t} = 2e^{-2t},$$

which is never zero. That means $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental set of solutions. Therefore, by the principle of superposition, the general solution is

$$\mathbf{x} = C_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

where C_1 and C_2 are arbitrary constants. The direction field is the two-dimensional vector field obtained from the system of equations.

$$\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4x_1 - 3x_2 \\ 8x_1 - 6x_2 \end{pmatrix} \quad \Rightarrow \quad \langle 4x_1 - 3x_2, 8x_1 - 6x_2 \rangle$$

The direction field is shown below in red, and the possible trajectories are shown below in blue.