

Problem 8

In each of Problems 7 and 8:

- Find the general solution of the given system of equations.
- Draw a direction field and a few of the trajectories. In each of these problems, the coefficient matrix has a zero eigenvalue. As a result, the pattern of trajectories is different from those in the examples in the text.

$$\mathbf{x}' = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \mathbf{x}$$

Solution

Because this is a constant-coefficient homogeneous linear system, it's expected to have solutions of the form $\mathbf{x} = e^{\lambda t} \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ has constant elements.

$$\lambda e^{\lambda t} \boldsymbol{\xi} = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} e^{\lambda t} \boldsymbol{\xi}$$

Divide both sides by $e^{\lambda t}$.

$$\lambda \boldsymbol{\xi} = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \boldsymbol{\xi}$$

This is now an eigenvalue problem.

$$\begin{aligned} \left\{ \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right\} \boldsymbol{\xi} &= \mathbf{0} \\ \begin{pmatrix} 3-\lambda & 6 \\ -1 & -2-\lambda \end{pmatrix} \boldsymbol{\xi} &= \mathbf{0} \end{aligned} \tag{1}$$

The eigenvalues satisfy

$$\det \begin{pmatrix} 3-\lambda & 6 \\ -1 & -2-\lambda \end{pmatrix} = 0.$$

Evaluate the determinant and solve for λ .

$$(3-\lambda)(-2-\lambda) + 6 = 0$$

$$-\lambda + \lambda^2 = 0$$

$$\lambda(\lambda - 1) = 0$$

$$\lambda = \{0, 1\}$$

Let

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = 0.$$

Substitute these two eigenvalues into equation (1) to determine the corresponding eigenvectors.

$$\begin{aligned}
 \begin{pmatrix} 3 - \lambda_1 & 6 \\ -1 & -2 - \lambda_1 \end{pmatrix} \boldsymbol{\xi}_1 &= \mathbf{0} & \begin{pmatrix} 3 - \lambda_2 & 6 \\ -1 & -2 - \lambda_2 \end{pmatrix} \boldsymbol{\xi}_2 &= \mathbf{0} \\
 \begin{pmatrix} 2 & 6 \\ -1 & -3 \end{pmatrix} \boldsymbol{\xi}_1 &= \mathbf{0} & \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \boldsymbol{\xi}_2 &= \mathbf{0} \\
 2\xi_1 + 6\xi_2 &= 0 & 3\xi_1 + 6\xi_2 &= 0 \\
 \xi_1 &= -3\xi_2 & \xi_1 &= -2\xi_2 \\
 \boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -3\xi_2 \\ \xi_2 \end{pmatrix} = \xi_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} & & \boldsymbol{\xi}_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -2\xi_2 \\ \xi_2 \end{pmatrix} = \xi_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}
 \end{aligned}$$

Two solutions to the system are then

$$\mathbf{x}_1 = e^{\lambda_1 t} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3e^t \\ e^t \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{\lambda_2 t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2e^t \\ e^t \end{pmatrix}.$$

The Wronskian of these two functions is

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} -3e^t & -2 \\ e^t & 1 \end{vmatrix} = -3e^t + 2e^t = -e^t,$$

which is never zero. That means \mathbf{x}_1 and \mathbf{x}_2 form a fundamental set of solutions. Therefore, by the principle of superposition, the general solution is

$$\mathbf{x} = C_1 e^t \begin{pmatrix} -3 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$

where C_1 and C_2 are arbitrary constants. The direction field is the two-dimensional vector field obtained from the system of equations.

$$\mathbf{x}' = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3x_1 + 6x_2 \\ -x_1 - 2x_2 \end{pmatrix} \Rightarrow \langle 3x_1 + 6x_2, -x_1 - 2x_2 \rangle$$

The direction field is shown below in red, and the possible trajectories are shown below in blue.

