Problem 9

In each of Problems 9 through 14, find the general solution of the given system of equations.

$$\mathbf{x}' = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \mathbf{x}$$

Solution

Because this is a constant-coefficient homogeneous linear system, it's expected to have solutions of the form $\mathbf{x} = e^{\lambda t}\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ has constant elements.

$$\lambda e^{\lambda t}\boldsymbol{\xi} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} e^{\lambda t}\boldsymbol{\xi}$$

Divide both sides by $e^{\lambda t}$.

$$\lambda\boldsymbol{\xi} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \boldsymbol{\xi}$$

This is now an eigenvalue problem.

$$\left\{ \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right\} \boldsymbol{\xi} = \mathbf{0}$$

$$\begin{pmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{pmatrix} \boldsymbol{\xi} = \mathbf{0} \tag{1}$$

The eigenvalues satisfy

$$\det \begin{pmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{pmatrix} = 0.$$

Evaluate the determinant and solve for λ.

$$(1-\lambda)(1-\lambda) + i^2 = 0$$

$$-2\lambda + \lambda^2 = 0$$

$$\lambda(\lambda - 2) = 0$$

$$\lambda = \{0, 2\}$$

Let

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 2.$$

Substitute these two eigenvalues into equation (1) to determine the corresponding eigenvectors.

$$\begin{pmatrix} 1-\lambda_1 & i \\ -i & 1-\lambda_1 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0}$$

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0}$$

$$\xi_1 + i\xi_2 = 0$$

$$\xi_1 = -i\xi_2$$

$$\boldsymbol{\xi}_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -i\xi_2 \\ \xi_2 \end{pmatrix} = \xi_2 \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1-\lambda_2 & i \\ -i & 1-\lambda_2 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0}$$

$$\begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0}$$

$$-\xi_1 + i\xi_2 = 0$$

$$\xi_1 = i\xi_2$$

$$\boldsymbol{\xi}_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} i\xi_2 \\ \xi_2 \end{pmatrix} = \xi_2 \begin{pmatrix} i \\ 1 \end{pmatrix}$$

Two solutions to the system are then

$$\mathbf{x}_1 = e^{\lambda_1 t}\begin{pmatrix}-i\\1\end{pmatrix} = \begin{pmatrix}-i\\1\end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{\lambda_2 t}\begin{pmatrix}i\\1\end{pmatrix} = \begin{pmatrix}ie^{2t}\\e^{2t}\end{pmatrix}.$$

Therefore, by the principle of superposition, the general solution is

$$\mathbf{x} = C_1\begin{pmatrix}-i\\1\end{pmatrix} + C_2e^{2t}\begin{pmatrix}i\\1\end{pmatrix},$$

where C_1 and C_2 are arbitrary constants.