Exercise 1

Find the square roots of \((a) \, 2i; \, (b) \, 1 - \sqrt{3}i\) and express them in rectangular coordinates.

\[\text{Ans. } (a) \pm (1 + i); \quad (b) \pm \frac{\sqrt{3} - i}{\sqrt{2}}.\]

Solution

For a nonzero complex number \(z = re^{i(\Theta + 2\pi k)}\), its square roots are

\[z^{1/2} = \left[re^{i(\Theta + 2\pi k)}\right]^{1/2} = r^{1/2} \exp\left(i \frac{\Theta + 2\pi k}{2}\right), \quad k = 0, 1.\]

Part (a)

The magnitude of \(2i\) is \(r = 2\), and the principal argument is \(\Theta = \pi/2\).

\[(2i)^{1/2} = 2^{1/2} \exp\left(i \frac{\pi + 2\pi k}{2}\right), \quad k = 0, 1\]

The first root \((k = 0)\) is

\[(2i)^{1/2} = 2^{1/2} e^{i\pi/4} = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) = \sqrt{2} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}\right) = 1 + i,\]

and the second root \((k = 1)\) is

\[(2i)^{1/2} = 2^{1/2} e^{5\pi/4} = \sqrt{2} \left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right) = \sqrt{2} \left(-\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}\right) = -1 - i.\]

\[\begin{align*}
\text{Im } z & \quad \bullet \quad z = 2i \\
\text{Re } z & \quad \bullet \quad z = 1 + i \\
\text{Im } z & \quad \bullet \quad z = -1 - i
\end{align*}\]
Part (b)

The magnitude and principal argument of $1 - \sqrt{3}i$ are respectively

$$r = \sqrt{1^2 + (-\sqrt{3})^2} = 2 \quad \text{and} \quad \Theta = \tan^{-1} \frac{-\sqrt{3}}{1} = -\frac{\pi}{3},$$

so

$$(1 - \sqrt{3}i)^{1/2} = 2^{1/2} \exp \left( i \frac{-\pi}{3} + \frac{2\pi k}{2} \right), \quad k = 0, 1.$$

The first root ($k = 0$) is

$$(1 - \sqrt{3}i)^{1/2} = 2^{1/2} e^{-i\pi/6} = \sqrt{2} \left( \cos \frac{\pi}{6} - i \sin \frac{\pi}{6} \right) = \sqrt{2} \left( \frac{\sqrt{3}}{2} - i \frac{1}{2} \right) = \frac{1}{\sqrt{2}} (\sqrt{3} - i),$$

and the second root ($k = 1$) is

$$(1 - \sqrt{3}i)^{1/2} = 2^{1/2} e^{i5\pi/6} = \sqrt{2} \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) = \sqrt{2} \left( -\frac{\sqrt{3}}{2} + i \frac{1}{2} \right) = \frac{1}{\sqrt{2}} (-\sqrt{3} - i).$$