**Exercise 2**

In each case, find all the roots in rectangular coordinates, exhibit them as vertices of certain squares, and point out which is the principal root:

\[(a) \ (−16)^{1/4}; \quad (b) \ (−8 − 8\sqrt{3}i)^{1/4}.\]

*Ans. (a) $±\sqrt{2}(1 + i), \ ±\sqrt{2}(1 − i); \quad (b) \ ±(\sqrt{3} − i), \ ±(1 + \sqrt{3}i).$*

**Solution**

For a nonzero complex number $z = re^{i(\Theta + 2\pi k)}$, its fourth roots are

\[z^{1/4} = \left[re^{i(\Theta + 2\pi k)}\right]^{1/4} = r^{1/4} \exp\left(i\frac{\Theta + 2\pi k}{4}\right), \quad k = 0, 1, 2, 3.\]

**Part (a)**

The magnitude of $−16$ is $16$, and the principal argument is $\pi$.

\[(-16)^{1/4} = 16^{1/4} \exp\left(i\frac{\pi + 2\pi k}{4}\right), \quad k = 0, 1, 2, 3\]

The first, or principal, root ($k = 0$) is

\[(-16)^{1/4} = 16^{1/4} e^{i\pi/4} = 2 \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) = 2 \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}\right) = \sqrt{2}(1 + i),\]

the second root ($k = 1$) is

\[(-16)^{1/4} = 16^{1/4} e^{i3\pi/4} = 2 \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right) = 2 \left(-\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}\right) = \sqrt{2}(-1 + i),\]

the third root ($k = 2$) is

\[(-16)^{1/4} = 16^{1/4} e^{i5\pi/4} = 2 \left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right) = 2 \left(-\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}\right) = \sqrt{2}(-1 − i),\]

and the fourth root ($k = 3$) is

\[(-16)^{1/4} = 16^{1/4} e^{i7\pi/4} = 2 \left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}\right) = 2 \left(\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}\right) = \sqrt{2}(1 − i).\]
Part (b)

The magnitude and principal argument of \(-8 - 8\sqrt{3}i\) are respectively

\[
r = \sqrt{(-8)^2 + (-8\sqrt{3})^2} = 16 \quad \text{and} \quad \Theta = \tan^{-1} \frac{-8\sqrt{3}}{-8} = -\frac{2\pi}{3},
\]

so

\[
(-8 - 8\sqrt{3}i)^{1/4} = 16^{1/4} \exp \left( i \frac{-2\pi}{3} + \frac{2\pi k}{4} \right), \quad k = 0, 1, 2, 3.
\]

The first, or principal, root (\(k = 0\)) is

\[
(-8 - 8\sqrt{3}i)^{1/4} = 16^{1/4} e^{-i\pi/6} = 2 \left( \cos \frac{\pi}{6} - i \sin \frac{\pi}{6} \right) = 2 \left( \frac{\sqrt{3}}{2} - i \frac{1}{2} \right) = \sqrt{3} - i,
\]

the second root (\(k = 1\)) is

\[
(-8 - 8\sqrt{3}i)^{1/4} = 16^{1/4} e^{i\pi/3} = 2 \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) = 2 \left( \frac{1}{2} + i \frac{\sqrt{3}}{2} \right) = 1 + \sqrt{3}i,
\]

the third root (\(k = 2\)) is

\[
(-8 - 8\sqrt{3}i)^{1/4} = 16^{1/4} e^{i5\pi/6} = 2 \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) = 2 \left( -\frac{\sqrt{3}}{2} + i \frac{1}{2} \right) = -\sqrt{3} + i,
\]

and the fourth root (\(k = 3\)) is

\[
(-8 - 8\sqrt{3}i)^{1/4} = 16^{1/4} e^{i4\pi/3} = 2 \left( \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right) = 2 \left( -\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) = -1 - \sqrt{3}i.
\]