Exercise 4

According to Sec. 9, the three cube roots of a nonzero complex number $z_0$ can be written $c_0$, $c_0\omega^3$, $c_0\omega^2$ where $c_0$ is the principal cube root of $z_0$ and

$$\omega = \exp\left(i\frac{2\pi}{3}\right) = \frac{-1 + \sqrt{3}i}{2}.$$ 

Show that if $z_0 = -4\sqrt{2} + 4\sqrt{2}i$, then $c_0 = \sqrt{2}(1 + i)$ and the other two cube roots are, in rectangular form, the numbers

$$c_0\omega_3 = \frac{-(\sqrt{3} + 1) + (\sqrt{3} - 1)i}{\sqrt{2}}, \quad c_0\omega_2^2 = \frac{(\sqrt{3} - 1) - (\sqrt{3} + 1)i}{\sqrt{2}}.$$

Solution

For a nonzero complex number $z = re^{i(\Theta + 2\pi k)}$, its third roots are

$$z^{1/3} = \left[r e^{i(\Theta + 2\pi k)}\right]^{1/3} = r^{1/3} \exp\left(i\frac{\Theta + 2\pi k}{3}\right), \quad k = 0, 1, 2.$$ 

The magnitude and principal argument of $z_0 = -4\sqrt{2} + 4\sqrt{2}i$ are respectively

$$r = \sqrt{(-4\sqrt{2})^2 + (4\sqrt{2})^2} = 8 \quad \text{and} \quad \Theta = \tan^{-1}\frac{4\sqrt{2}}{-4\sqrt{2}} + \pi = \frac{3\pi}{4},$$

so

$$z_0^{1/3} = 8^{1/3} \exp\left(i\frac{3\pi + 2\pi k}{3}\right) = 2e^{i\pi/4} \exp\left(i\frac{2\pi k}{3}\right), \quad k = 0, 1, 2.$$ 

The first, or principal, root ($k = 0$) is

$$z_0^{1/3} = c_0 = 2e^{i\pi/4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}(1 + i),$$

the second root ($k = 1$) is

$$z_0^{1/3} = c_0\omega_3 = 2e^{i\pi/4} \exp\left(i\frac{2\pi}{3}\right) = 2e^{i\pi/12} = 2\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right) = 2\left(-\frac{\sqrt{3} + 1}{2\sqrt{2}} + i\frac{\sqrt{3} - 1}{2\sqrt{2}}\right) = \frac{-(\sqrt{3} + 1) + (\sqrt{3} - 1)i}{\sqrt{2}},$$

and the third root ($k = 2$) is

$$z_0^{1/3} = c_0\omega_2^3 = 2e^{i\pi/4} \exp\left(i\frac{4\pi}{3}\right) = 2e^{i\pi/12} = 2\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right) = 2\left(\frac{\sqrt{3} - 1}{2\sqrt{2}} - i\frac{\sqrt{3} + 1}{2\sqrt{2}}\right) = \frac{(\sqrt{3} - 1) - (\sqrt{3} + 1)i}{\sqrt{2}}.$$