Exercise 4

According to Sec. 9, the three cube roots of a nonzero complex number z_0 can be written c_0, $c_0\omega_3$, $c_0\omega_3^2$ where c_0 is the principal cube root of z_0 and

$$\omega_3 = \exp\left(i\frac{2\pi}{3}\right) = \frac{-1+\sqrt{3}i}{2}.$$

Show that if $z_0 = -4\sqrt{2} + 4\sqrt{2}i$, then $c_0 = \sqrt{2}(1+i)$ and the other two cube roots are, in rectangular form, the numbers

$$c_0\omega_3 = \frac{-(\sqrt{3}+1)+(\sqrt{3}-1)i}{\sqrt{2}}, \quad c_0\omega_3^2 = \frac{(\sqrt{3}-1)-(\sqrt{3}+1)i}{\sqrt{2}}.$$

Solution

For a nonzero complex number $z = re^{i(\Theta+2\pi k)}$, its third roots are

$$z^{1/3} = \left[re^{i(\Theta+2\pi k)}\right]^{1/3} = r^{1/3}\exp\left(i\frac{\Theta+2\pi k}{3}\right), \quad k = 0, 1, 2.$$

The magnitude and principal argument of $z_0 = -4\sqrt{2} + 4\sqrt{2}i$ are respectively

$$r = \sqrt{(-4\sqrt{2})^2 + (4\sqrt{2})^2} = 8 \quad \text{and} \quad \Theta = \tan^{-1}\frac{4\sqrt{2}}{-4\sqrt{2}} + \pi = \frac{3\pi}{4},$$

so

$$z_0^{1/3} = 8^{1/3}\exp\left(i\frac{\frac{3\pi}{4}+2\pi k}{3}\right) = 2e^{i\pi/4}\exp\left(i\frac{2\pi k}{3}\right), \quad k = 0, 1, 2.$$

The first, or principal, root ($k = 0$) is

$$z_0^{1/3} = c_0 = 2e^{i\pi/4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}(1+i),$$

the second root ($k = 1$) is

$$\begin{aligned} z_0^{1/3} = c_0\omega_3 = 2e^{i\pi/4}\exp\left(i\frac{2\pi}{3}\right) = 2e^{i11\pi/12} = 2\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right) &= 2\left(-\frac{\sqrt{3}+1}{2\sqrt{2}} + i\frac{\sqrt{3}-1}{2\sqrt{2}}\right) \\ &= \frac{-(\sqrt{3}+1)+(\sqrt{3}-1)i}{\sqrt{2}}, \end{aligned}$$

and the third root ($k = 2$) is

$$\begin{aligned} z_0^{1/3} = c_0\omega_3^2 = 2e^{i\pi/4}\exp\left(i\frac{4\pi}{3}\right) = 2e^{i19\pi/12} = 2\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right) &= 2\left(\frac{\sqrt{3}-1}{2\sqrt{2}} - i\frac{\sqrt{3}+1}{2\sqrt{2}}\right) \\ &= \frac{(\sqrt{3}-1)-(\sqrt{3}+1)i}{\sqrt{2}}. \end{aligned}$$