Exercise 6

Find the four roots of the equation $z^4 + 4 = 0$ and use them to factor $z^4 + 4$ into quadratic factors with real coefficients.

$$\textit{Ans.}\ (z^2 + 2z + 2)(z^2 - 2z + 2).$$

[The wording of this question was taken from the 7th edition.]

Solution

Our aim here is to find the fourth roots of -4. For a nonzero complex number $z = re^{i(\Theta + 2\pi k)}$, its fourth roots are

$$z^{1/4} = \left[re^{i(\Theta+2\pi k)}\right]^{1/4} = r^{1/4} \exp\left(i\frac{\Theta + 2\pi k}{4}\right), \quad k = 0, 1, 2, 3.$$

The magnitude and principal argument of -4 are respectively $r = 4$ and $\Theta = \pi$, so

$$z^{1/4} = 4^{1/4} \exp\left(i\frac{\pi + 2\pi k}{4}\right), \quad k = 0, 1, 2, 3.$$

The first root $(k = 0)$ is

$$z^{1/4} = 4^{1/4}e^{i\pi/4} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 1 + i,$$

the second root $(k = 1)$ is

$$z^{1/4} = 4^{1/4}e^{i3\pi/4} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = -1 + i,$$

the third root $(k = 2)$ is

$$z^{1/4} = 4^{1/4}e^{i5\pi/4} = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = -1 - i,$$

and the fourth root $(k = 3)$ is

$$z^{1/4} = 4^{1/4}e^{i7\pi/4} = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = 1 - i.$$

Consequently, the left side of $z^4 + 4 = 0$ can be factored like so.

$$[z - (1 + i)][z - (-1 + i)][z - (-1 - i)][z - (1 - i)] = 0.$$

Multiply the first and fourth terms to obtain a quadratic factor with real coefficients. Multiply the second and third terms to get the second quadratic factor.

$$(z^2 + 2z + 2)(z^2 - 2z + 2) = 0$$