Exercise 9

Let $z = re^{i\theta}$ be a nonzero complex number and n a negative integer ($n = -1, -2, \ldots$). Then define $z^{1/n}$ by means of the equation $z^{1/n} = (z^{-1})^{1/m}$ where $m = -n$. By showing that the m values of $(z^{1/m})^{-1}$ and $(z^{-1})^{1/m}$ are the same, verify that $z^{1/n} = (z^{1/m})^{-1}$. (Compare with Exercise 7, Sec. 8.)

Solution

If z is a nonzero complex number, then it can be represented in polar form as $z = re^{i\theta}$. Start by evaluating $(z^{1/m})^{-1}$, where m is a positive integer.

$$\begin{aligned}
(z^{1/m})^{-1} &= \frac{1}{z^{1/m}} \\
&= \frac{1}{(re^{i\theta})^{1/m}} \\
&= \frac{1}{r^{1/m}e^{i\theta/m}} \\
&= \frac{1}{r^{1/m}} \cdot \frac{1}{e^{i\theta/m}} \\
&= r^{-1/m} \cdot \frac{e^{i0}}{e^{i\theta/m}} \\
&= r^{-1/m} \cdot e^{i0-i\theta/m} \\
&= r^{-1/m}e^{-i\theta/m}
\end{aligned}$$

Now evaluate $(z^{-1})^{1/m}$.

$$\begin{aligned}
(z^{-1})^{1/m} &= \left(\frac{1}{z}\right)^{1/m} \\
&= \left(\frac{1}{re^{i\theta}}\right)^{1/m} \\
&= \left(\frac{1}{r} \cdot \frac{1}{e^{i\theta}}\right)^{1/m} \\
&= \left(r^{-1} \cdot \frac{e^{i0}}{e^{i\theta}}\right)^{1/m} \\
&= \left(r^{-1} \cdot e^{i0-i\theta}\right)^{1/m} \\
&= \left(r^{-1} \cdot e^{-i\theta}\right)^{1/m} \\
&= r^{-1/m}e^{-i\theta/m}
\end{aligned}$$

Since $(z^{1/m})^{-1} = (z^{-1})^{1/m}$, the definition of $z^{1/n}$ can also be written as $z^{1/n} = (z^{1/m})^{-1}$.