Exercise 10

Prove that

(a) \( z \) is real if and only if \( \bar{z} = z \);

(b) \( z \) is either real or pure imaginary if and only if \( z^2 = \bar{z}^2 \).

Solution

Part (a)

Suppose that \( z \) is real. Then \( z = x + i0 = x \) and \( \bar{z} = x - i0 = x \). Thus, \( \bar{z} = z \).

Suppose that \( \bar{z} = z \). Then \( x - iy = x + iy \), or \( -iy = iy \). This equation is only satisfied if \( y = 0 \). The imaginary component is zero, so \( z \) is real.

Therefore, \( z \) is real if and only if \( \bar{z} = z \).

Part (b)

Suppose that \( z \) is real. Then \( \bar{z} = z \) from part (a). Square both sides to get \( z^2 = \bar{z}^2 \).

Suppose that \( z \) is purely imaginary. Then \( z = 0 + iy = iy \) and \( \bar{z} = 0 - iy = -iy \). Then \( z^2 = -y^2 = \bar{z}^2 \).

Suppose that \( z^2 = \bar{z}^2 \). Then

\[
(x - iy)^2 = (x + iy)^2 \\
x^2 - 2ixy - y^2 = x^2 + 2ixy - y^2 \\
-2ixy = 2ixy \\
x = 0 \quad \text{or} \quad y = 0.
\]

Thus, \( z = x + iy \) is either real or purely imaginary.

Therefore, \( z \) is either real or purely imaginary if and only if \( z^2 = \bar{z}^2 \).