Exercise 11

(a) Use the binomial formula (Sec. 3) and de Moivre's formula (Sec. 7) to write

$$\cos n\theta + i\sin n\theta = \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta (i\sin\theta)^k \qquad (n = 0, 1, 2, \ldots).$$

Then define the integer m by means of the equations

$$m = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n\text{is odd} \end{cases}$$

and use the above summation to show that [compare with Exercise 10(a)]

$$\cos n\theta = \sum_{k=0}^{m} \binom{n}{2k} (-1)^k \cos^{n-2k}\theta \sin^{2k}\theta \qquad (n = 0, 1, 2, \ldots).$$

(b) Write $x = \cos\theta$ in the final summation in part (a) to show that it becomes a polynomial

$$T_n(x) = \sum_{k=0}^{m} \binom{n}{2k} (-1)^k x^{n-2k}(1-x^2)^k$$

of degree n ($n = 0, 1, 2, \ldots$) in the variable x.*

[TYPO: A space is needed between "n" and "is." Oddly, this mistake wasn't in the 7th edition.]

Solution

Part (a)

The binomial theorem states that for two complex numbers, z_1 and z_2,

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k} \qquad (n = 1, 2, \ldots),$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Set $z_1 = i\sin\theta$ and $z_2 = \cos\theta$.

$$(i\sin\theta + \cos\theta)^n = \sum_{k=0}^{n} \binom{n}{k} (i\sin\theta)^k (\cos\theta)^{n-k}$$

Apply de Moivre's theorem on the left.

$$\cos n\theta + i\sin n\theta = \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta (i\sin\theta)^k \tag{1}$$

*These are called Chebyshev polynomials and are prominent in approximation theory.

Take the complex conjugate of both sides.

$$\cos n\theta - i\sin n\theta = \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta(-i\sin\theta)^k \tag{2}$$

Add the respective sides of equations (1) and (2).

$$\begin{aligned} 2\cos n\theta &= \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta(i\sin\theta)^k + \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta(-i\sin\theta)^k \\ &= \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta(i\sin\theta)^k + \sum_{k=0}^{n}(-1)^k \binom{n}{k} \cos^{n-k}\theta(i\sin\theta)^k \\ &= \sum_{k=0}^{n}[1+(-1)^k] \binom{n}{k} \cos^{n-k}\theta(i\sin\theta)^k \end{aligned} \tag{3}$$

Notice that if k is odd, the summand is zero. The series can thus be simplified (that is, made to converge faster) by substituting $k = 2p$. Suppose first that n is even.

$$2\cos n\theta = \sum_{2p=0}^{n} (2) \binom{n}{2p} \cos^{n-2p}\theta(i\sin\theta)^{2p}$$

Consequently,

$$\cos n\theta = \sum_{p=0}^{\frac{n}{2}} \binom{n}{2p} \cos^{n-2p}\theta(i\sin\theta)^{2p} \quad \text{if } n \text{ is even.} \tag{4}$$

Suppose secondly that n is odd. Then equation (3) can be written as

$$2\cos n\theta = \sum_{k=0}^{n-1}[1+(-1)^k] \binom{n}{k} \cos^{n-k}\theta(i\sin\theta)^k + \underbrace{[1+(-1)^n] \binom{n}{n} \cos^{n-n}\theta(i\sin\theta)^n}_{=0}.$$

Substitute $k = 2p$ as before to simplify the series.

$$2\cos n\theta = \sum_{2p=0}^{n-1} (2) \binom{n}{2p} \cos^{n-2p}\theta(i\sin\theta)^{2p}$$

Consequently,

$$\cos n\theta = \sum_{p=0}^{\frac{n-1}{2}} \binom{n}{2p} \cos^{n-2p}\theta(i\sin\theta)^{2p} \quad \text{if } n \text{ is odd.} \tag{5}$$

If we define m to be

$$m = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases},$$

then equations (4) and (5) can be combined like so. (p is just a dummy variable and can be replaced with k.)

$$\cos n\theta = \sum_{p=0}^{m} \binom{n}{2p} (-1)^p \cos^{n-2p}\theta \sin^{2p}\theta$$

Part (b)

The result of part (a) can be written as

$$\begin{aligned}\cos n\theta &= \sum_{p=0}^{m} \binom{n}{2p} (-1)^p (\cos\theta)^{n-2p} (\sin^2\theta)^p \\ &= \sum_{p=0}^{m} \binom{n}{2p} (-1)^p (\cos\theta)^{n-2p} (1-\cos^2\theta)^p.\end{aligned}$$

Let $x = \cos\theta$. Then $\theta = \cos^{-1} x$, and the right side becomes a polynomial in x.

$$\cos\left(n\cos^{-1} x\right) = \sum_{p=0}^{m} \binom{n}{2p} (-1)^p x^{n-2p} (1-x^2)^p$$

$T_n(x) = \cos\left(n\cos^{-1} x\right)$ are known as the Chebyshev polynomials.