Exercise 3

Use mathematical induction to show that

$$e^{i\theta_1}e^{i\theta_2}\cdots e^{i\theta_n} = e^{i(\theta_1+\theta_2+\cdots+\theta_n)} \qquad (n = 2, 3, \ldots).$$

Solution

Start by showing that the result is true in the base case $n = 2$.

$$\begin{aligned} e^{i\theta_1}e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \\ &= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) \\ &= e^{i(\theta_1+\theta_2)} \end{aligned}$$

Now assume the inductive hypothesis,

$$e^{i\theta_1}e^{i\theta_2}\cdots e^{i\theta_k} = e^{i(\theta_1+\theta_2+\cdots+\theta_k)} \qquad (k = 2, 3, \ldots).$$

The aim is to show that

$$e^{i\theta_1}e^{i\theta_2}\cdots e^{i\theta_k}e^{i\theta_{k+1}} = e^{i(\theta_1+\theta_2+\cdots+\theta_k+\theta_{k+1})} \qquad (k = 2, 3, \ldots).$$

We have

$$\begin{aligned} e^{i\theta_1}e^{i\theta_2}\cdots e^{i\theta_k}e^{i\theta_{k+1}} &= (e^{i\theta_1}e^{i\theta_2}\cdots e^{i\theta_k})e^{i\theta_{k+1}} \\ &= [e^{i(\theta_1+\theta_2+\cdots+\theta_k)}]e^{i\theta_{k+1}} \\ &= [\cos(\theta_1+\theta_2+\cdots+\theta_k) + i\sin(\theta_1+\theta_2+\cdots+\theta_k)](\cos\theta_{k+1} + i\sin\theta_{k+1}) \\ &= \cos(\theta_1+\theta_2+\cdots+\theta_k)\cos\theta_{k+1} - \sin(\theta_1+\theta_2+\cdots+\theta_k)\sin\theta_{k+1} \\ &\qquad + i[\cos(\theta_1+\theta_2+\cdots+\theta_k)\sin\theta_{k+1} + \sin(\theta_1+\theta_2+\cdots+\theta_k)\cos\theta_{k+1}] \\ &= \cos[(\theta_1+\theta_2+\cdots+\theta_k)+\theta_{k+1}] + i\sin[(\theta_1+\theta_2+\cdots+\theta_k)+\theta_{k+1}] \\ &= \cos(\theta_1+\theta_2+\cdots+\theta_k+\theta_{k+1}) + i\sin(\theta_1+\theta_2+\cdots+\theta_k+\theta_{k+1}) \\ &= e^{i(\theta_1+\theta_2+\cdots+\theta_k+\theta_{k+1})}. \end{aligned}$$

Therefore, by mathematical induction,

$$e^{i\theta_1}e^{i\theta_2}\cdots e^{i\theta_n} = e^{i(\theta_1+\theta_2+\cdots+\theta_n)} \qquad (n = 2, 3, \ldots).$$