

## Exercise 5

By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

$$\begin{aligned} (a) \quad & i(1 - \sqrt{3}i)(\sqrt{3} + i) = 2(1 + \sqrt{3}i); & (b) \quad & 5i/(2 + i) = 1 + 2i; \\ (c) \quad & (-1 + i)^7 = -8(1 + i); & (d) \quad & (1 + \sqrt{3}i)^{-10} = 2^{-11}(-1 + \sqrt{3}i). \end{aligned}$$

### Solution

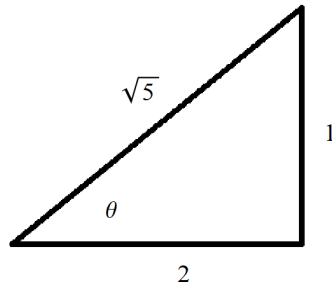
#### Part (a)

$$\begin{aligned} i(1 - \sqrt{3}i)(\sqrt{3} + i) &= \left( e^{i\pi/2} \right) \left[ \sqrt{1^2 + (-\sqrt{3})^2} \exp \left( i \tan^{-1} \frac{-\sqrt{3}}{1} \right) \right] \left[ \sqrt{(\sqrt{3})^2 + 1^2} \exp \left( i \tan^{-1} \frac{1}{\sqrt{3}} \right) \right] \\ &= e^{i\pi/2} (2e^{-i\pi/3}) (2e^{i\pi/6}) \\ &= 4e^{i(\pi/2 - \pi/3 + \pi/6)} \\ &= 4e^{i\pi/3} \\ &= 4 \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \\ &= 4 \left( \frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \\ &= 2(1 + \sqrt{3}i) \end{aligned}$$

#### Part (b)

$$\begin{aligned} \frac{5i}{2 + i} &= \frac{5e^{i\pi/2}}{\sqrt{2^2 + 1^2} \exp \left( i \tan^{-1} \frac{1}{2} \right)} \\ &= \sqrt{5} \exp \left[ i \left( \frac{\pi}{2} - \tan^{-1} \frac{1}{2} \right) \right] \\ &= \sqrt{5} \left[ \cos \left( \frac{\pi}{2} - \tan^{-1} \frac{1}{2} \right) + i \sin \left( \frac{\pi}{2} - \tan^{-1} \frac{1}{2} \right) \right] \\ &= \sqrt{5} \left( \underbrace{\cos \frac{\pi}{2}}_{=0} \cos \tan^{-1} \frac{1}{2} + \underbrace{\sin \frac{\pi}{2}}_{=1} \sin \tan^{-1} \frac{1}{2} \right) + i \left( \underbrace{\sin \frac{\pi}{2}}_{=1} \cos \tan^{-1} \frac{1}{2} - \underbrace{\cos \frac{\pi}{2}}_{=0} \sin \tan^{-1} \frac{1}{2} \right) \\ &= \sqrt{5} \left( \sin \tan^{-1} \frac{1}{2} + i \cos \tan^{-1} \frac{1}{2} \right) \end{aligned}$$

Let  $\theta = \tan^{-1}(1/2)$  and draw the implied right triangle to determine the quantities in parentheses.



So then

$$\begin{aligned} \frac{5i}{2+i} &= \sqrt{5}(\sin \theta + i \cos \theta) \\ &= \sqrt{5} \left( \frac{1}{\sqrt{5}} + i \frac{2}{\sqrt{5}} \right) \\ &= 1 + 2i. \end{aligned}$$

Part (c)

$$\begin{aligned} (-1+i)^7 &= \left\{ \sqrt{(-1)^2+1^2} \exp \left[ i \left( \tan^{-1} \frac{1}{-1} + \pi \right) \right] \right\}^7 \\ &= \left[ \sqrt{2} e^{i(-\pi/4+\pi)} \right]^7 \\ &= \left( \sqrt{2} e^{3i\pi/4} \right)^7 \\ &= \sqrt{2^7} e^{21i\pi/4} \\ &= 8\sqrt{2} \left( \cos \frac{21\pi}{4} + i \sin \frac{21\pi}{4} \right) \\ &= 8\sqrt{2} \left( -\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right) \\ &= -8(1+i) \end{aligned}$$

Part (d)

$$\begin{aligned} (1+\sqrt{3}i)^{-10} &= \left[ \sqrt{1^2+(\sqrt{3})^2} \exp \left( i \tan^{-1} \frac{\sqrt{3}}{1} \right) \right]^{-10} \\ &= (2e^{i\pi/3})^{-10} \\ &= 2^{-10} e^{-10i\pi/3} \\ &= 2^{-10} \left( \cos \frac{10\pi}{3} - i \sin \frac{10\pi}{3} \right) \\ &= 2^{-10} \left( -\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \\ &= 2^{-11} (-1 + \sqrt{3}i) \end{aligned}$$