Exercise 6

Show that if \( \text{Re} z_1 > 0 \) and \( \text{Re} z_2 > 0 \), then
\[
\text{Arg}(z_1 z_2) = \text{Arg} z_1 + \text{Arg} z_2,
\]
where principal arguments are used.

Solution

Suppose that \( \text{Re} z_1 > 0 \) and \( \text{Re} z_2 > 0 \). Then \( z_1 = r_1 e^{i\theta_1} \) and \( z_2 = r_2 e^{i\theta_2} \) lie in the first or fourth quadrants:
\[
-\frac{\pi}{2} < \theta_1 + 2n_1 \pi < \frac{\pi}{2}
\]
(1)
and
\[
-\frac{\pi}{2} < \theta_2 + 2n_2 \pi < \frac{\pi}{2}
\]
(2)
where \( n_1 = 0, \pm 1, \pm 2, \ldots \) and \( n_2 = 0, \pm 1, \pm 2, \ldots \). We have
\[
\text{arg} z_1 = \text{arg} r_1 e^{i\theta_1} = \theta_1 + 2n_1 \pi.
\]
Since this quantity on the right is between \( -\pi/2 \) and \( \pi/2 \), \( \text{arg} z_1 \) can be replaced by the principal argument \( \text{Arg} z_1 \). (The requirement to use \( \text{Arg} z \) is that \( -\pi < \text{Arg} z \leq \pi \).)
\[
\text{arg} z_1 = \text{Arg} z_1
\]
Similarly, we have
\[
\text{arg} z_2 = \text{arg} r_2 e^{i\theta_2} = \theta_2 + 2n_2 \pi.
\]
Since this quantity on the right is between \( -\pi/2 \) and \( \pi/2 \), \( \text{arg} z_2 \) can be replaced by the principal argument \( \text{Arg} z_2 \).
\[
\text{arg} z_2 = \text{Arg} z_2
\]
Add each of the sides of inequalities (1) and (2).
\[
-\pi < \theta_1 + \theta_2 + 2n_1 \pi + 2n_2 \pi < \pi
\]
(3)
We also have
\[
\text{arg}(z_1 z_2) = \text{arg} z_1 + \text{arg} z_2
\]
\[
= \text{arg} r_1 e^{i\theta_1} + \text{arg} r_2 e^{i\theta_2}
\]
\[
= (\theta_1 + 2n_1 \pi) + (\theta_2 + 2n_2 \pi)
\]
\[
= \theta_1 + \theta_2 + 2n_1 \pi + 2n_2 \pi.
\]
Since
\[
-\pi < \text{arg}(z_1 z_2) < \pi,
\]
\( \text{arg}(z_1 z_2) \) can be replaced by the principal argument \( \text{Arg}(z_1 z_2) \). Therefore, because
\[
\text{arg}(z_1 z_2) = \text{arg} z_1 + \text{arg} z_2,
\]
\[
\text{Arg}(z_1 z_2) = \text{Arg} z_1 + \text{Arg} z_2
\]
for the case that \( z_1 \) and \( z_2 \) have positive real components.