Exercise 9

Establish the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1)$$

and then use it to derive *Lagrange's trigonometric identity:*

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \qquad (0 < \theta < 2\pi).$$

Suggestion: As for the first identity, write $S = 1 + z + z^2 + \cdots + z^n$ and consider the difference $S - zS$. To derive the second identity, write $z = e^{i\theta}$ in the first one.

Solution

Let $S = 1 + z + z^2 + \cdots + z^n$. Then

$$\begin{aligned} S - zS &= 1 + z + z^2 + \cdots + z^n - z(1 + z + z^2 + \cdots + z^n) \\ &= 1 + z - z + z^2 - z^2 + \cdots + z^n - z^n - z^{n+1} \\ &= 1 - z^{n+1}. \end{aligned}$$

Factor the left side

$$S(1 - z) = 1 - z^{n+1}$$

and then divide both sides by $1 - z$.

$$S = \frac{1 - z^{n+1}}{1 - z}$$

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

Substitute $z = e^{i\theta}$ into this equation.

$$\begin{aligned} 1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} &= \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \\ &= \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \cdot \frac{e^{-i\theta/2}}{e^{-i\theta/2}} \\ &= \frac{e^{-i\theta/2} - e^{i\left(n+\frac{1}{2}\right)\theta}}{e^{-i\theta/2} - e^{i\theta/2}} \\ &= \frac{-\frac{e^{i\left(n+\frac{1}{2}\right)\theta} - e^{-i\theta/2}}{2i}}{-\frac{e^{i\theta/2} - e^{-i\theta/2}}{2i}} \\ &= \frac{-\frac{e^{i\left(n+\frac{1}{2}\right)\theta} - e^{-i\theta/2}}{2i}}{-\sin\frac{\theta}{2}} \\ &= \frac{e^{i\left(n+\frac{1}{2}\right)\theta} - e^{-i\theta/2}}{2i\sin\frac{\theta}{2}} \end{aligned}$$

Continue the simplification of the right side by using Euler's formula.

$$\begin{aligned}
1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} &= \frac{\cos\left(n+\frac{1}{2}\right)\theta + i\sin\left(n+\frac{1}{2}\right)\theta - \left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)}{2i\sin\frac{\theta}{2}} \\
&= \frac{i\sin\frac{\theta}{2} + i\sin\left(n+\frac{1}{2}\right)\theta - \cos\frac{\theta}{2} + \cos\left(n+\frac{1}{2}\right)\theta}{2i\sin\frac{\theta}{2}} \\
&= \frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} + \frac{-\cos\frac{\theta}{2} + \cos\left(n+\frac{1}{2}\right)\theta}{2i\sin\frac{\theta}{2}} \\
&= \frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} + i\frac{\cos\frac{\theta}{2} - \cos\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}
\end{aligned}$$

On the left side use de Moivre's formula, $(\cos\theta + i\sin\theta)^m = e^{im\theta} = \cos m\theta + i\sin m\theta$, which holds for any integer m.

$$1 + (\cos\theta + i\sin\theta) + (\cos 2\theta + i\sin 2\theta) + \cdots + (\cos n\theta + i\sin n\theta) = \frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} + i\frac{\cos\frac{\theta}{2} - \cos\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}$$

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta + i(\sin\theta + \sin 2\theta + \cdots + \sin n\theta) = \frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} + i\frac{\cos\frac{\theta}{2} - \cos\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}$$

Matching the real part of each side yields Lagrange's trigonometric identity.

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left[(2n+1)\theta/2\right]}{2\sin(\theta/2)}$$