Exercise 3

Show that

(a) \( \text{Res} \_{z=z_n} (z \sec z) = (-1)^{n+1}z_n \) where \( z_n = \frac{\pi}{2} + n\pi \) \( (n = 0, \pm 1, \pm 2, \ldots) \);

(b) \( \text{Res} \_{z=z_n} (\tanh z) = 1 \) where \( z_n = \left( \frac{\pi}{2} + n\pi \right)i \) \( (n = 0, \pm 1, \pm 2, \ldots) \).

Solution

Part (a)

The residue at \( z = z_n \) can be calculated by

\[
\text{Res} \_{z=z_n} z \sec z = \text{Res} \_{z=z_n} \frac{z}{\cos z} = \frac{p_1(z_n)}{q_1'(z_n)},
\]

where \( p_1(z) \) is set to be the function in the numerator and \( q_1(z) \) is set to be the function in the denominator.

\[
p_1(z) = z \quad \Rightarrow \quad p_1(z_n) = z_n
\]

\[
q_1(z) = \cos z \quad \rightarrow \quad q_1'(z) = -\sin z \quad \Rightarrow \quad q_1'(z_n) = -\sin \left( \frac{\pi}{2} + n\pi \right) = -\cos n\pi = (-1)^{n+1}
\]

Therefore,

\[
\text{Res} \_{z=z_n} z \sec z = (-1)^{n+1}z_n.
\]

Part (b)

The residue at \( z = z_n \) can be calculated by

\[
\text{Res} \_{z=z_n} \tanh z = \text{Res} \_{z=z_n} \frac{\sinh z}{\cosh z} = \frac{p_2(z_n)}{q_2'(z_n)},
\]

where \( p_2(z) \) is set to be the function in the numerator and \( q_2(z) \) is set to be the function in the denominator.

\[
p_1(z) = \sinh z \quad \Rightarrow \quad p_1(z_n) = \sinh \left( \frac{\pi}{2} + n\pi \right)i
\]

\[
q_1(z) = \cosh z \quad \rightarrow \quad q_1'(z) = \sinh z \quad \Rightarrow \quad q_1'(z_n) = \sinh \left( \frac{\pi}{2} + n\pi \right)i
\]

All that’s left to do is to check that \( p_1(z_n) = q_1'(z_n) \neq 0 \).

\[
\sinh \left( \frac{\pi}{2} + n\pi \right)i = i \sin \left( \frac{\pi}{2} + n\pi \right) = i \cos n\pi = i(-1)^n
\]

Therefore,

\[
\text{Res} \_{z=z_n} \tanh z = 1.
\]