Exercise 3

Show that

(a) $\operatorname*{Res}_{z=z_n}(z \sec z) = (-1)^{n+1} z_n$ where $z_n = \dfrac{\pi}{2} + n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $\operatorname*{Res}_{z=z_n}(\tanh z) = 1$ where $z_n = \left(\dfrac{\pi}{2} + n\pi\right) i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

Solution

Part (a)

The residue at $z = z_n$ can be calculated by

$$\operatorname*{Res}_{z=z_n} z \sec z = \operatorname*{Res}_{z=z_n} \frac{z}{\cos z} = \frac{p_1(z_n)}{q_1'(z_n)},$$

where $p_1(z)$ is set to be the function in the numerator and $q_1(z)$ is set to be the function in the denominator.

$$\begin{aligned} p_1(z) &= z & & \Rightarrow \quad p_1(z_n) = z_n \\ q_1(z) &= \cos z \quad \rightarrow \quad q_1'(z) = -\sin z & & \Rightarrow \quad q_1'(z_n) = -\sin\left(\frac{\pi}{2} + n\pi\right) = -\cos n\pi = (-1)^{n+1} \end{aligned}$$

Therefore,

$$\operatorname*{Res}_{z=z_n} z \sec z = (-1)^{n+1} z_n.$$

Part (b)

The residue at $z = z_n$ can be calculated by

$$\operatorname*{Res}_{z=z_n} \tanh z = \operatorname*{Res}_{z=z_n} \frac{\sinh z}{\cosh z} = \frac{p_2(z_n)}{q_2'(z_n)},$$

where $p_2(z)$ is set to be the function in the numerator and $q_2(z)$ is set to be the function in the denominator.

$$\begin{aligned} p_1(z) &= \sinh z & & \Rightarrow \quad p_1(z_n) = \sinh\left[\left(\frac{\pi}{2} + n\pi\right) i\right] \\ q_1(z) &= \cosh z \quad \rightarrow \quad q_1'(z) = \sinh z & & \Rightarrow \quad q_1'(z_n) = \sinh\left[\left(\frac{\pi}{2} + n\pi\right) i\right] \end{aligned}$$

All that's left to do is to check that $p_1(z_n) = q_1'(z_n) \neq 0$.

$$\sinh\left[\left(\frac{\pi}{2} + n\pi\right) i\right] = i \sin\left(\frac{\pi}{2} + n\pi\right) = i \cos n\pi = i(-1)^n$$

Therefore,

$$\operatorname*{Res}_{z=z_n} \tanh z = 1.$$