Exercise 4

Let C denote the positively oriented circle $|z| = 2$ and evaluate the integral

$$\text{(a)} \int_C \tan z \, dz; \qquad \text{(b)} \int_C \frac{dz}{\sinh 2z}.$$

Ans. (a) $-4\pi i$; (b) $-\pi i$.

Solution

Part (a)

The singularities of the integrand,

$$\tan z = \frac{\sin z}{\cos z},$$

occur where the denominator is zero.

$$\cos z = 0 \quad \rightarrow \quad z = \frac{1}{2}(2n-1)\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

The ones we care about are those that lie within the circle $|z| = 2$: $z = -\pi/2$ and $z = \pi/2$. According to Cauchy's residue theorem, the closed loop integral over this circle is equal to $2\pi i$ times the sum of the residues inside it.

$$\oint_C \tan z \, dz = 2\pi i \left(\operatorname*{Res}_{z=-\pi/2} \tan z + \operatorname*{Res}_{z=\pi/2} \tan z \right)$$

The residues at $z = \pm\pi/2$ can be calculated by

$$\operatorname*{Res}_{z=\pm\pi/2} \tan z = \operatorname*{Res}_{z=\pm\pi/2} \frac{\sin z}{\cos z} = \frac{p(\pm\pi/2)}{q(\pm\pi/2)},$$

where $p(z)$ is set to be the function in the numerator and $q(z)$ is set to be the function in the denominator.

$$p(z) = \sin z \qquad \Rightarrow \qquad p\left(\pm\frac{\pi}{2}\right) = \sin\left(\pm\frac{\pi}{2}\right) = \pm 1$$

$$q(z) = \cos z \quad \rightarrow \quad q'(z) = -\sin z \quad \Rightarrow \quad q'\left(\pm\frac{\pi}{2}\right) = -\sin\left(\pm\frac{\pi}{2}\right) = \mp 1$$

So then

$$\begin{aligned} \oint_C \tan z \, dz &= 2\pi i \left(\frac{-1}{1} + \frac{1}{-1} \right) \\ &= 2\pi i(-2). \end{aligned}$$

Therefore,

$$\oint_C \tan z \, dz = -4\pi i.$$

Part (b)

The singularities of the integrand,

$$\frac{1}{\sinh 2z},$$

occur where the denominator is zero.

$$\begin{aligned} \sinh 2z &= 0 \\ -i\sin 2iz &= 0 \\ 2iz &= n\pi \quad \rightarrow \quad z = -\frac{in\pi}{2}, \quad n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

The ones we care about are those that lie within the circle $|z| = 2$: $z = 0$, $z = -i\pi/2$, and $z = i\pi/2$. According to Cauchy's residue theorem, the closed loop integral over this circle is equal to $2\pi i$ times the sum of the residues inside it.

$$\oint_C \frac{dz}{\sinh 2z} = 2\pi i \left(\operatorname*{Res}_{z=-i\pi/2} \frac{1}{\sinh 2z} + \operatorname*{Res}_{z=0} \frac{1}{\sinh 2z} + \operatorname*{Res}_{z=i\pi/2} \frac{1}{\sinh 2z} \right)$$

The residues at $z = \pm i\pi/2$ and $z = 0$ can be calculated by

$$\begin{aligned} \operatorname*{Res}_{z=\pm i\pi/2} \frac{1}{\sinh 2z} &= \frac{p(\pm i\pi/2)}{q(\pm i\pi/2)} \\ \operatorname*{Res}_{z=0} \frac{1}{\sinh 2z} &= \frac{p(0)}{q(0)}, \end{aligned}$$

where $p(z)$ is set to be the function in the numerator and $q(z)$ is set to be the function in the denominator.

$$p(z) = 1 \qquad \Rightarrow \qquad \begin{cases} p\left(\pm \frac{i\pi}{2}\right) = 1 \\ p(0) = 1 \end{cases}$$

$$q(z) = \sinh 2z \quad \rightarrow \quad q'(z) = 2\cosh 2z \quad \Rightarrow \quad \begin{cases} q'\left(\pm \frac{i\pi}{2}\right) = 2\cosh(\pm i\pi) = 2\cos\pi = -2 \\ q'(0) = 2\cosh 0 = 2 \end{cases}$$

So then

$$\begin{aligned} \oint_C \frac{dz}{\sinh 2z} &= 2\pi i \left(\frac{1}{-2} + \frac{1}{2} + \frac{1}{-2} \right) \\ &= 2\pi i \left(-\frac{1}{2} \right). \end{aligned}$$

Therefore,

$$\oint_C \frac{dz}{\sinh 2z} = -\pi i.$$