Exercise 10

The integration formula

\[
\int_0^\infty \frac{dx}{[(x^2-a)^2+1]^2} = \frac{\pi}{8\sqrt{2}A^3}\left[(2a^2+3)\sqrt{A+a} + a\sqrt{A-a}\right],
\]

where \(a\) is any real number and \(A = \sqrt{a^2+1}\), arises in the theory of case-hardening of steel by means of radio-frequency heating.\(^*\) Follow the steps below to derive it.

(a) Point out why the four zeros of the polynomial

\[q(z) = (z^2-a)^2 + 1\]

are the square roots of the numbers \(a \pm i\). Then, using the fact that the numbers

\[z_0 = \frac{1}{\sqrt{2}}(\sqrt{A+a} + i\sqrt{A-a})\]

and \(-z_0\) are the square roots of \(a + i\) (Exercise 5, Sec. 10), verify that \(\pm z_0\) are the square roots of \(a - i\) and hence that \(z_0\) and \(-z_0\) are the only zeros of \(q(z)\) in the upper half plane \(\text{Im}\,z \geq 0\).

(b) Using the method derived in Exercise 7, Sec. 76, and keeping in mind that \(z_0^2 = a + i\) for purposes of simplification, show that the point \(z_0\) in part (a) is a pole of order 2 of the function \(f(z) = 1/[q(z)]^2\) and that the residue \(B_1\) at \(z_0\) can be written

\[B_1 = -\frac{q''(z_0)}{[q'(z_0)]^3} = \frac{a - i(2a^2 + 3)}{16A^2z_0}.\]

After observing that \(q'(-z) = -\overline{q'(z)}\) and \(q''(-z) = \overline{q''(z)}\), use the same method to show that the point \(-\overline{z_0}\) in part (a) is also a pole of order 2 of the function \(f(z)\), with residue

\[B_2 = \left\{ \frac{q''(\overline{z_0})}{[q'(\overline{z_0})]^3} \right\} = -\overline{B_1}.\]

Then obtain the expression

\[B_1 + B_2 = \frac{1}{8A^2i} \text{Im} \left[ \frac{-a + i(2a^2 + 3)}{z_0} \right]\]

for the sum of the residues.

(c) Refer to part (a) and show that \(|q(z)| \geq (R - |z_0|)^4\) if \(|z| = R\), where \(R > |z_0|\). Then, with the aid of the final result in part (b), complete the derivation of the integration formula.

\(^*\)See pp. 359-364 of the book by Brown, Hoyler, and Bierwirth that is listed in Appendix 1.

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