Exercise 3

Use residues to evaluate the improper integrals in Exercises 1 through 5.

\[
\int_0^\infty \frac{dx}{x^4 + 1}.
\]

Ans. \(\pi/(2\sqrt{2})\).

Solution

The integrand is an even function of \(x\), so the interval of integration can be extended to \((-\infty, \infty)\) as long as the integral is divided by 2.

\[
\int_0^\infty \frac{dx}{x^4 + 1} = \int_{-\infty}^{\infty} \frac{dx}{2(x^4 + 1)}
\]

In order to evaluate the integral, consider the corresponding function in the complex plane,

\[f(z) = \frac{1}{2(z^4 + 1)},\]

and the contour in Fig. 93. Singularities occur where the denominator is equal to zero.

\[2(z^4 + 1) = 0\]
\[z^4 + 1 = 0\]

\[z = \sqrt[4]{-1} \exp \left[ i \left( \frac{\pi + 2k\pi}{4} \right) \right], \quad k = 0, 1, 2, 3 \rightarrow
\left\{
\begin{array}{l}
z_1 = e^{i\pi/4} = \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \\
z_2 = e^{i3\pi/4} = -\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \\
z_3 = e^{i5\pi/4} = -\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \\
z_4 = e^{i7\pi/4} = \frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}
\end{array}
\right.

The singular points of interest to us are the ones that lie within the closed contour, \(z = z_1\) and \(z = z_2\).

Figure 1: This is Fig. 93 with the singularities at \(z = z_1\) and \(z = z_2\) marked.

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According to Cauchy’s residue theorem, the integral of \(1/[2(z^4 + 1)]\) around the closed contour is equal to \(2\pi i\) times the sum of the residues at the enclosed singularities.

\[
\oint_C \frac{dz}{2(z^4 + 1)} = 2\pi i \left[ \text{Res}_{z=z_1} \frac{1}{2(z^4 + 1)} + \text{Res}_{z=z_2} \frac{1}{2(z^4 + 1)} \right]
\]

This closed loop integral is the sum of two integrals, one over each arc in the loop.

\[
\int_L \frac{dz}{2(z^4 + 1)} + \int_{C_R} \frac{dz}{2(z^4 + 1)} = 2\pi i \left[ \text{Res}_{z=z_1} \frac{1}{2(z^4 + 1)} + \text{Res}_{z=z_2} \frac{1}{2(z^4 + 1)} \right]
\]

The parameterizations for the arcs are as follows.

\[\begin{align*}
L : & \quad z = r, \quad r = -R \quad \rightarrow \quad r = R \\
C_R : & \quad z = Re^{i\theta}, \quad \theta = 0 \quad \rightarrow \quad \theta = \pi
\end{align*}\]

As a result,

\[
\int_{-R}^{R} \frac{dr}{2(r^4 + 1)} + \int_{C_R} \frac{dz}{2(z^4 + 1)} = 2\pi i \left[ \text{Res}_{z=z_1} \frac{1}{2(z^4 + 1)} + \text{Res}_{z=z_2} \frac{1}{2(z^4 + 1)} \right].
\]

Take the limit now as \(R \to \infty\). The integral over \(C_R\) consequently tends to zero. Proof for this statement will be given at the end.

\[
\int_{-\infty}^{\infty} \frac{dr}{2(r^4 + 1)} = 2\pi i \left[ \text{Res}_{z=z_1} \frac{1}{2(z^4 + 1)} + \text{Res}_{z=z_2} \frac{1}{2(z^4 + 1)} \right]
\]

The denominator can be written as \(2(z^4 + 1) = 2(z - z_1)(z - z_2)(z - z_3)(z - z_4)\). From this we see that the multiplicities of the \(z - z_1\) and \(z - z_2\) factors are both 1. The residues at \(z = z_1\) and \(z = z_2\) can then be calculated by

\[
\begin{align*}
\text{Res}_{z=z_1} \frac{1}{2(z^4 + 1)} & = \phi_1(z_1) \\
\text{Res}_{z=z_2} \frac{1}{2(z^4 + 1)} & = \phi_2(z_2),
\end{align*}
\]

where \(\phi_1(z)\) and \(\phi_2(z)\) are equal to \(f(z)\) without the \(z - z_1\) and \(z - z_2\) factors, respectively.

\[
\begin{align*}
\phi_1(z) & = \frac{1}{2(z - z_2)(z - z_3)(z - z_4)} \Rightarrow \phi_1(z_1) = \frac{1}{2(\sqrt{2}[\sqrt{2}(1+i)][i\sqrt{2}])} = -\frac{1}{8\sqrt{2}}(1+i) \\
\phi_2(z) & = \frac{1}{2(z - z_1)(z - z_3)(z - z_4)} \Rightarrow \phi_2(z_2) = \frac{1}{2(-\sqrt{2})(i\sqrt{2})[\sqrt{2}(-1+i)]} = \frac{1}{8\sqrt{2}}(1-i)
\end{align*}
\]

So then

\[
\begin{align*}
\text{Res}_{z=z_1} \frac{1}{2(z^4 + 1)} & = -\frac{1}{8\sqrt{2}}(1+i) \\
\text{Res}_{z=z_2} \frac{1}{2(z^4 + 1)} & = \frac{1}{8\sqrt{2}}(1-i)
\end{align*}
\]
\[
\int_{-\infty}^{\infty} \frac{dr}{2(r^4 + 1)} = 2\pi i \left[ -\frac{1}{8\sqrt{2}}(1 + i) + \frac{1}{8\sqrt{2}}(1 - i) \right]
\]
\[
= 2\pi i \left( \frac{-i}{4\sqrt{2}} \right)
\]
\[
= \frac{\pi}{2\sqrt{2}}.
\]

Therefore, changing the dummy integration variable to \( x \),

\[
\int_{0}^{\infty} \frac{dx}{x^4 + 1} = \frac{\pi}{2\sqrt{2}}.
\]
The Integral Over $C_R$

Our aim here is to show that the integral over $C_R$ tends to zero in the limit as $R \to \infty$. The parameterization of the semicircular arc in Fig. 93 is $z = Re^{i\theta}$, where $\theta$ goes from 0 to $\pi$.

\[
\int_{C_R} \frac{dz}{2(z^4 + 1)} = \int_{0}^{\pi} \frac{Re^{i\theta} d\theta}{2[(Re^{i\theta})^4 + 1]} = \int_{0}^{\pi} \frac{Re^{i\theta} d\theta}{R^4e^{4i\theta} + 1}
\]

Now consider the integral’s magnitude.

\[
\left| \int_{C_R} \frac{dz}{2(z^4 + 1)} \right| = \left| \int_{0}^{\pi} \frac{Re^{i\theta} d\theta}{R^4e^{4i\theta} + 1} \right| 
\leq \int_{0}^{\pi} \left| \frac{Re^{i\theta}}{R^4e^{4i\theta} + 1} \right| d\theta 
\leq \int_{0}^{\pi} \frac{|R|}{|R^4e^{4i\theta} + 1|} d\theta 
= \int_{0}^{\pi} \frac{R}{R^4e^{4i\theta} + 1} d\theta 
= \int_{0}^{\pi} \frac{R}{R^4 - 1} d\theta 
= \frac{\pi}{2} \frac{R}{R^4 - 1}
\]

Now take the limit of both sides as $R \to \infty$.

\[
\lim_{R \to \infty} \left| \int_{C_R} \frac{dz}{2(z^4 + 1)} \right| \leq \lim_{R \to \infty} \frac{\pi}{2} \frac{R}{R^4 - 1} 
= \lim_{R \to \infty} \frac{\pi}{2R^3} \frac{1}{1 - \frac{1}{R^4}}
\]

The limit on the right side is zero.

\[
\lim_{R \to \infty} \left| \int_{C_R} \frac{dz}{2(z^4 + 1)} \right| \leq 0
\]

The magnitude of a number cannot be negative.

\[
\lim_{R \to \infty} \left| \int_{C_R} \frac{dz}{2(z^4 + 1)} \right| = 0
\]

The only number that has a magnitude of zero is zero. Therefore,

\[
\lim_{R \to \infty} \int_{C_R} \frac{dz}{2(z^4 + 1)} = 0.
\]