Exercise 8

Use a residue and the contour shown in Fig. 95, where \( R > 1 \), to establish the integration formula

\[
\int_0^\infty \frac{dx}{x^3 + 1} = \frac{2\pi}{3\sqrt{3}}.
\]

Solution

In order to evaluate the integral, consider the corresponding function in the complex plane,

\[
f(z) = \frac{1}{z^3 + 1},
\]

and the contour in Fig. 95. Singularities occur where the denominator is equal to zero.

\[
z^3 + 1 = 0
\]

\[
z = \sqrt[3]{1} \exp \left[ i \left( \frac{\pi + 2k\pi}{3} \right) \right], \quad k = 0, 1, 2 \quad \rightarrow \quad \begin{cases} 
  z_1 = e^{i\pi/3} = \frac{1}{2} + i \frac{\sqrt{3}}{2} \\
  z_2 = e^{i\pi} = -1 \\
  z_3 = e^{i5\pi/3} = \frac{1}{2} - i \frac{\sqrt{3}}{2}
\end{cases}
\]

The singular point of interest to us is the one that lies within the closed contour, \( z = e^{i\pi/3} \).

According to Cauchy’s residue theorem, the integral of \( 1/(z^3 + 1) \) around the closed contour is equal to \( 2\pi i \) times the sum of the residues at the enclosed singularities.

\[
\oint_C \frac{dz}{z^3 + 1} = 2\pi i \frac{\text{Res} \frac{1}{z^3 + 1}}{z = z_1}
\]

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This closed loop integral is the sum of three integrals, one over each arc in the loop.

\[
\int_{L_1} \frac{dz}{z^3 + 1} + \int_{L_2} \frac{dz}{z^3 + 1} + \int_{C_R} \frac{dz}{z^3 + 1} = 2\pi i \text{Res}_{z=z_1} \frac{1}{z^3 + 1}
\]

The parameterizations for the arcs are as follows.

- \(L_1: z = re^{i\theta}, \quad r = 0 \rightarrow r = R\)
- \(L_2: z = re^{i2\pi/3}, \quad r = R \rightarrow r = 0\)
- \(C_R: z = Re^{i\theta}, \quad \theta = 0 \rightarrow \theta = \frac{2\pi}{3}\)

As a result,

\[
2\pi i \text{Res}_{z=z_1} \frac{1}{z^3 + 1} = \int_0^R \frac{dr}{r(e^{i\theta})^3 + 1} + \int_0^R \frac{dr}{(r e^{i2\pi/3})^3 + 1} + \int_{C_R} \frac{dz}{z^3 + 1}
\]

\[
= \left(1 - e^{i2\pi/3}\right) \int_0^R \frac{dr}{r^3 + 1} + \int_{C_R} \frac{dz}{z^3 + 1}
\]

Take the limit now as \(R \to \infty\). The integral over \(C_R\) consequently tends to zero. Proof for this statement will be given at the end.

\[
(1 - e^{i2\pi/3}) \int_0^\infty \frac{dr}{r^3 + 1} = 2\pi i \text{Res}_{z=z_1} \frac{1}{z^3 + 1}
\]

The denominator can be written as \(z^3 + 1 = (z - z_1)(z - z_2)(z - z_3)\). From this we see that the multiplicity of the \(z - z_1\) factor is 1. The residue at \(z = z_1\) can then be calculated by

\[
\text{Res}_{z=z_1} \frac{1}{z^3 + 1} = \phi(z_1),
\]

where \(\phi(z)\) is equal to \(f(z)\) without the \(z - z_1\) factor.

\[
\phi(z) = \frac{1}{(z - z_2)(z - z_3)} \Rightarrow \phi(z_1) = \frac{1}{(e^{i\pi/3} + 1)(i\sqrt{3})} = \frac{1}{\left(\frac{3}{2} + i\frac{\sqrt{3}}{2}\right)(i\sqrt{3})} = \frac{3 - i\sqrt{3}}{3(i\sqrt{3})}
\]

So then

\[
\text{Res}_{z=z_1} \frac{1}{z^3 + 1} = \frac{3 - i\sqrt{3}}{3i\sqrt{3}}
\]

and

\[
(1 - e^{i2\pi/3}) \int_0^\infty \frac{dr}{r^3 + 1} = 2\pi i \left(\frac{3}{2} - i\frac{\sqrt{3}}{2}\right)
\]

\[
\left(\frac{3}{2} - i\frac{\sqrt{3}}{2}\right) \int_0^\infty \frac{dr}{r^3 + 1} = \frac{2\pi}{3\sqrt{3}} \left(\frac{3}{2} - i\frac{\sqrt{3}}{2}\right).
\]

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Cancel the terms in parentheses.

\[ \int_0^\infty \frac{dr}{r^3 + 1} = \frac{2\pi}{3\sqrt{3}} \]

Therefore, changing the dummy integration variable to \( x \),

\[
\int_0^\infty \frac{dx}{x^3 + 1} = \frac{2\pi}{3\sqrt{3}}.
\]
The Integral Over $C_R$

Our aim here is to show that the integral over $C_R$ tends to zero in the limit as $R \to \infty$. The parameterization of the circular arc in Fig. 93 is $z = Re^{i\theta}$, where $\theta$ goes from 0 to $2\pi/3$.

$$\int_{C_R} \frac{dz}{z^3 + 1} = \int_0^{2\pi/3} \frac{Re^{i\theta} d\theta}{(Re^{i\theta})^3 + 1} = \int_0^{2\pi/3} \frac{Re^{i\theta} d\theta}{R^3 e^{i\theta} + 1}$$

Now consider the integral’s magnitude.

$$\left| \int_{C_R} \frac{dz}{z^3 + 1} \right| = \left| \int_0^{2\pi/3} \frac{R e^{i\theta} d\theta}{R^3 e^{i3\theta} + 1} \right| \leq \int_0^{2\pi/3} \left| \frac{Re^{i\theta}}{R^3 e^{i3\theta} + 1} \right| d\theta$$

$$= \int_0^{2\pi/3} \frac{Re^{i\theta}}{|R^3 e^{i3\theta} + 1|} d\theta \leq \int_0^{2\pi/3} \frac{R}{|R^3 e^{i3\theta} + 1| - 1} d\theta = \int_0^{2\pi/3} \frac{R}{R^3 - 1} d\theta = \frac{2\pi}{3} \frac{R}{R^3 - 1}$$

Now take the limit of both sides as $R \to \infty$.

$$\lim_{R \to \infty} \left| \int_{C_R} \frac{dz}{z^3 + 1} \right| \leq \lim_{R \to \infty} \frac{2\pi}{3} \frac{R}{R^3 - 1} = \lim_{R \to \infty} \frac{2\pi}{3R^2} \frac{1}{1 - \frac{1}{R^3}}$$

The limit on the right side is zero.

$$\lim_{R \to \infty} \left| \int_{C_R} \frac{dz}{z^3 + 1} \right| \leq 0$$

The magnitude of a number cannot be negative.

$$\lim_{R \to \infty} \left| \int_{C_R} \frac{dz}{z^3 + 1} \right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R \to \infty} \int_{C_R} \frac{dz}{z^3 + 1} = 0.$$