Exercise 9

Let \( m \) and \( n \) be integers, where \( 0 \leq m < n \). Follow the steps below to derive the integration formula
\[
\int_0^\infty \frac{x^{2m}}{x^{2n} + 1} \, dx = \frac{\pi}{2n} \csc \left( \frac{2m + 1}{2n} \pi \right).
\]

(a) Show that the zeros of the polynomial \( z^{2n} + 1 \) lying above the real axis are
\[
c_k = \exp \left[ \frac{i(2k + 1)\pi}{2n} \right] \quad (k = 0, 1, 2, \ldots, n - 1)
\]
and that there are none on that axis.

(b) With the aid of Theorem 2 in Sec. 76, show that
\[
\text{Res}_{z=c_k} \frac{z^{2m}}{z^{2n} + 1} = -\frac{1}{2n} e^{i(2k+1)\alpha} \quad (k = 0, 1, 2, \ldots, n - 1)
\]
where \( c_k \) are the zeros found in part (a) and
\[
\alpha = \frac{2m + 1}{2n} \pi.
\]

Then use the summation formula
\[
\sum_{k=0}^{n-1} c_k = \frac{1 - z^n}{1 - z} \quad (z \neq 1)
\]
(see Exercise 9, Sec. 8) to obtain the expression
\[
2\pi i \sum_{k=0}^{n-1} \text{Res}_{z=c_k} \frac{z^{2m}}{z^{2n} + 1} = \frac{\pi}{n \sin \alpha}.
\]

(c) Use the final result in part (b) to complete the derivation of the integration formula.

Solution

The integrand is an even function of \( x \), so the interval of integration can be extended to \(( -\infty, \infty )\) as long as the integral is divided by 2.

\[
\int_0^\infty \frac{x^{2m}}{x^{2n} + 1} \, dx = \int_0^\infty \frac{(x^2)^m}{(x^2)^n + 1} \, dx = \int_{-\infty}^{\infty} \frac{x^{2m}}{2(x^{2n} + 1)} \, dx
\]

In order to evaluate the integral, consider the corresponding function in the complex plane,
\[
f(z) = \frac{z^{2m}}{2(z^{2n} + 1)},
\]
and the contour in Fig. 93. Singularities occur where the denominator is equal to zero.
\[
2(z^{2n} + 1) = 0
\]
\[
z^{2n} + 1 = 0
\]
\[
z = \sqrt[2n]{-1} \exp \left[ i \left( \frac{\pi + 2k\pi}{2n} \right) \right], \quad k = 0, 1, \ldots, 2n - 1
\]

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The singular points of interest to us are the ones that lie within the closed contour, that is, those with a positive imaginary component. Use Euler’s formula to write the exponential function in terms of sine and cosine.

\[ z = \cos \left( \frac{\pi + 2k\pi}{2n} \right) + i \sin \left( \frac{\pi + 2k\pi}{2n} \right), \quad k = 0, 1, \ldots, 2n - 1 \]

We require

\[ \sin \left( \frac{\pi + 2k\pi}{2n} \right) > 0, \quad k = 0, 1, \ldots, 2n - 1, \]

so the argument of sine must have a value between 0 and \( \pi \).

\[ 0 < \frac{\pi + 2k\pi}{2n} < \pi \]

\[ 0 < \frac{1 + 2k}{2n} < 1 \]

\[ 0 < 1 + 2k < 2n \]

The values of \( k \) that satisfy this inequality are \( k = 0, 1, \ldots, n - 1 \). Thus, the singular points that lie within the contour are

\[ z = z_k = \exp \left[ i \left( \frac{\pi + 2k\pi}{2n} \right) \right], \quad k = 0, 1, \ldots, n - 1. \]

According to Cauchy’s residue theorem, the integral of \( z^{2m}/[2(z^{2n} + 1)] \) around the closed contour is equal to \( 2\pi i \) times the sum of the residues at the enclosed singularities.

\[ \oint_C \frac{z^{2m}}{2(z^{2n} + 1)} \, dz = 2\pi i \sum_{k=0}^{n-1} \text{Res}_{z=z_k} \frac{z^{2m}}{2(z^{2n} + 1)} \]

This closed loop integral is the sum of two integrals, one over each arc in the loop.

\[ \int_L \frac{z^{2m}}{2(z^{2n} + 1)} \, dz + \int_{C_R} \frac{z^{2m}}{2(z^{2n} + 1)} \, dz = 2\pi i \sum_{k=0}^{n-1} \text{Res}_{z=z_k} \frac{z^{2m}}{2(z^{2n} + 1)} \]

The parameterizations for the arcs are as follows.

\[ L : \quad z = r, \quad r = -R \rightarrow r = R \]

\[ C_R : \quad z = R e^{i\theta}, \quad \theta = 0 \rightarrow \theta = \pi \]

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As a result,
\[ \int_{-R}^{R} \frac{r^{2m}}{2(r^{2n} + 1)} \, dr + \int_{C_R} \frac{z^{2m}}{2(z^{2n} + 1)} \, dz = 2\pi i \sum_{k=0}^{n-1} \text{Res}_{z=z_{2k}} \frac{z^{2m}}{2(z^{2n} + 1)}. \]

Take the limit now as \( R \to \infty \). The integral over \( C_R \) consequently tends to zero. Proof for this statement will be given at the end.

\[ \int_{-\infty}^{\infty} \frac{r^{2m}}{2(r^{2n} + 1)} \, dr = 2\pi i \sum_{k=0}^{n-1} \text{Res}_{z=z_k} \frac{z^{2m}}{2(z^{2n} + 1)} \]

The residue at \( z = z_k \) can be calculated by
\[
\text{Res}_{z=z_k} \frac{z^{2m}}{2(z^{2n} + 1)} = \frac{p(z_k)}{q'(z_k)},
\]
where \( p(z) \) and \( q(z) \) are equal to the numerator and denominator of \( f(z) \), respectively.

\[
p(z) = z^{2m} \quad \Rightarrow \quad p(z_k) = \exp \left[ i \left( \frac{\pi + 2k\pi}{2n} \right) 2m \right]
\]

\[
q(z) = 2(z^{2n} + 1) \quad \Rightarrow \quad q'(z) = 4nz^{2n-1} \quad \Rightarrow \quad q'(z_k) = 4n \exp \left[ i \left( \frac{\pi + 2k\pi}{2n} \right) (2n - 1) \right]
\]

So then
\[
\text{Res}_{z=z_k} \frac{z^{2m}}{2(z^{2n} + 1)} = \frac{\exp \left[ i \left( \frac{\pi + 2k\pi}{2n} \right) 2m \right]}{4n \exp \left[ i \left( \frac{\pi + 2k\pi}{2n} \right) (2n - 1) \right]}
\]

\[
= \frac{1}{4n} \exp \left[ i \left( \frac{\pi + 2k\pi}{2n} \right) \frac{(2m - 2n + 1)}{2n} \right]
\]

\[
= \frac{1}{4n} \exp \left[ i(2k + 1) \frac{2m + 1}{2n} \pi \right]
\]

\[
= \frac{1}{4n} \exp \left[ i(2k + 1) \frac{2m + 1}{2n} \pi \right] \exp [i(2k + 1)(-1)\pi]
\]

\[
= -\frac{1}{4n} \exp \left( i \frac{2m + 1}{n} \pi \right) \exp \left( -\frac{2m + 1}{2n} \pi \right)
\]

and
\[
\sum_{k=0}^{n-1} \text{Res}_{z=z_k} \frac{z^{2m}}{2(z^{2n} + 1)} = \sum_{k=0}^{n-1} \left\{ -\frac{1}{4n} \exp \left( i \frac{2m + 1}{n} \pi \right) \exp \left( i \frac{2m + 1}{2n} \pi \right) \right\}
\]

\[
= -\frac{1}{4n} \exp \left( i \frac{2m + 1}{2n} \pi \right) \sum_{k=0}^{n-1} \exp \left( i \frac{2m + 1}{n} \pi \right)
\]

\[
= -\frac{1}{4n} \exp \left( i \frac{2m + 1}{2n} \pi \right) \left[ \exp \left( i \frac{2m + 1}{n} \pi \right) \right]^{n}
\]

\[
= -\frac{1}{4n} \exp \left( i \frac{2m + 1}{2n} \pi \right) \frac{1 - \exp \left( i \frac{2m + 1}{n} \pi \right)^n}{1 - \exp \left( i \frac{2m + 1}{n} \pi \right)}
\]
\[
\sum_{k=0}^{n-1} \frac{z^{2m}}{2(z^{2n} + 1)} \text{Res}_{z=\pm 1} = -\frac{1}{4n} \exp\left(i\frac{2m+1}{2n} \pi \right) \frac{1 - \exp[i(2m+1)\pi]}{1 - \exp\left(i\frac{2m+1}{n} \pi \right)}
\]
\[
= -\frac{1}{4n} \exp\left(i\frac{2m+1}{2n} \pi \right) \frac{1 - (-1)}{1 - \exp\left(i\frac{2m+1}{n} \pi \right)}
\]
\[
= -\frac{1}{2n} \exp\left(i\frac{2m+1}{2n} \pi \right) \frac{1}{1 - \exp\left(i\frac{2m+1}{n} \pi \right)}
\]
\[
= -\frac{1}{2n} \left[ \frac{1}{\exp(-i\frac{2m+1}{2n} \pi) - \exp\left(i\frac{2m+1}{2n} \pi \right)} \right]
\]
\[
= -\frac{1}{2n} \left[ -\frac{1}{2i \sin\left(\frac{2m+1}{2n} \pi \right)} \right]
\]
\[
= \frac{1}{2n} \frac{1}{2i \sin\left(\frac{2m+1}{2n} \pi \right)}
\]

and

\[
\int_{-\infty}^{\infty} \frac{r^{2m}}{2(r^{2n} + 1)} \, dr = 2\pi i \left[ \frac{1}{2n} \frac{1}{2i \sin\left(\frac{2m+1}{2n} \pi \right)} \right]
\]
\[
= \frac{\pi}{2n \sin\left(\frac{2m+1}{2n} \pi \right)}
\]
\[
= \frac{\pi}{2n \csc\left(\frac{2m+1}{2n} \pi \right)}.
\]

Therefore, changing the dummy integration variable to \(x\),

\[
\int_{0}^{\infty} \frac{x^{2m}}{x^{2n} + 1} \, dx = \frac{\pi}{2n \csc\left(\frac{2m+1}{2n} \pi \right)}.
\]
The Integral Over $C_R$

Our aim here is to show that the integral over $C_R$ tends to zero in the limit as $R \to \infty$. The parameterization of the semicircular arc in Fig. 93 is $z = Re^{i\theta}$, where $\theta$ goes from 0 to $\pi$.

\[
\int_{C_R} \frac{z^{2m}}{2(z^{2n} + 1)} \, dz = \int_0^\pi \frac{(Re^{i\theta})^{2m}}{2[(Re^{i\theta})^{2n} + 1]} (Re^{i\theta} \, d\theta)
\]

\[
= \int_0^\pi \frac{R^{2m+1}e^{i\theta(2m+1)}}{R^{2n}e^{2n\theta} + 1} \frac{d\theta}{2}
\]

Now consider the integral’s magnitude.

\[
\left| \int_{C_R} \frac{z^{2m}}{2(z^{2n} + 1)} \, dz \right| = \left| \int_0^\pi \frac{R^{2m+1}e^{i\theta(2m+1)}}{R^{2n}e^{2n\theta} + 1} \frac{d\theta}{2} \right|
\]

\[
\leq \int_0^\pi \frac{|R^{2m+1}e^{i\theta(2m+1)}|}{|R^{2n}e^{2n\theta} + 1|} \frac{d\theta}{2}
\]

\[
= \int_0^\pi \frac{R^{2m+1}}{|R^{2n}e^{2n\theta} + 1|} \frac{d\theta}{2}
\]

\[
\leq \int_0^\pi \frac{R^{2m+1}}{|R^{2n}e^{2n\theta} - 1|} \frac{d\theta}{2}
\]

\[
= \int_0^\pi \frac{R^{2m+1}}{R^{2n} - 1} \frac{d\theta}{2}
\]

Now take the limit of both sides as $R \to \infty$.

\[
\lim_{R \to \infty} \left| \int_{C_R} \frac{z^{2m}}{2(z^{2n} + 1)} \, dz \right| \leq \lim_{R \to \infty} \frac{\pi}{2} \frac{R^{2m+1}}{R^{2n} - 1}
\]

\[
= \lim_{R \to \infty} \frac{\pi}{2} \frac{R^{2m+1}}{2R^{2n-2m-1} - 1 - \frac{1}{R^{2m}}}
\]

Since $n > m$ and $n$ and $m$ are integers, $2n - 2m - 1 > 0$, and the limit on the right side is zero.

\[
\lim_{R \to \infty} \left| \int_{C_R} \frac{z^{2m}}{2(z^{2n} + 1)} \, dz \right| \leq 0
\]

The magnitude of a number cannot be negative.

\[
\lim_{R \to \infty} \left| \int_{C_R} \frac{z^{2m}}{2(z^{2n} + 1)} \, dz \right| = 0
\]

The only number that has a magnitude of zero is zero. Therefore,

\[
\lim_{R \to \infty} \int_{C_R} \frac{z^{2m}}{2(z^{2n} + 1)} \, dz = 0.
\]