Exercise 9

Let m and n be integers, where $0 \leq m < n$. Follow the steps below to derive the integration formula

$$\int_0^\infty \frac{x^{2m}}{x^{2n}+1}\,dx = \frac{\pi}{2n}\csc\left(\frac{2m+1}{2n}\pi\right).$$

(a) Show that the zeros of the polynomial $z^{2n}+1$ lying above the real axis are

$$c_k = \exp\left[i\frac{(2k+1)\pi}{2n}\right] \qquad (k = 0, 1, 2, \ldots, n-1)$$

and that there are none on that axis.

(b) With the aid of Theorem 2 in Sec. 76, show that

$$\operatorname*{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = -\frac{1}{2n}e^{i(2k+1)\alpha} \qquad (k = 0, 1, 2, \ldots, n-1)$$

where c_k are the zeros found in part (a) and

$$\alpha = \frac{2m+1}{2n}\pi.$$

Then use the summation formula

$$\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z} \qquad (z \neq 1)$$

(see Exercise 9, Sec. 8) to obtain the expression

$$2\pi i \sum_{k=0}^{n-1} \operatorname*{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = \frac{\pi}{n\sin\alpha}.$$

(c) Use the final result in part (b) to complete the derivation of the integration formula.