

Exercise 6

Use residues to evaluate the improper integrals in Exercises 1 through 8.

$$\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4 + 4} dx \quad (a > 0).$$

$$\text{Ans. } \pi e^{-a} \cos a.$$

Solution

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{z^3 e^{iaz}}{z^4 + 4},$$

and the contour in Fig. 93. Singularities occur where the denominator is equal to zero.

$$z^4 + 4 = 0$$

$$z = \sqrt[4]{4} \exp \left[i \left(\frac{\pi + 2k\pi}{4} \right) \right], \quad k = 0, 1, 2, 3 \quad \rightarrow \quad \begin{cases} z_1 = \sqrt{2} e^{i\pi/4} = 1 + i \\ z_2 = \sqrt{2} e^{i3\pi/4} = -1 + i \\ z_3 = \sqrt{2} e^{i5\pi/4} = -1 - i \\ z_4 = \sqrt{2} e^{i7\pi/4} = 1 - i \end{cases}$$

The singular points of interest to us are the ones that lie within the closed contour, $z = z_1$ and $z = z_2$.

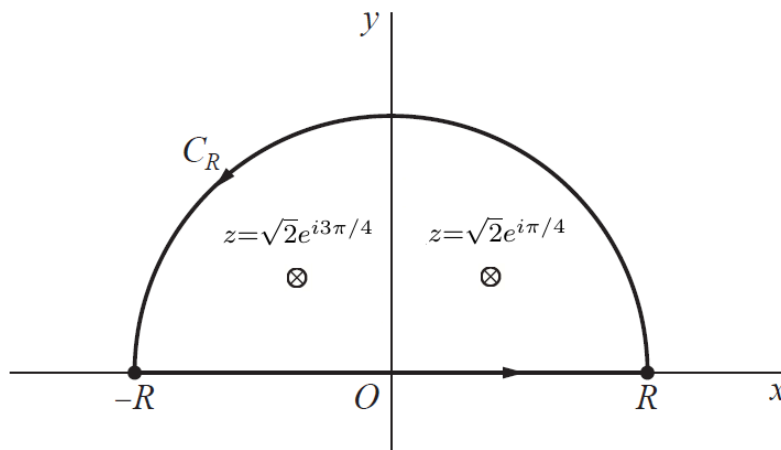


Figure 1: This is Fig. 93 with the singularities at $z = z_1$ and $z = z_2$ marked.

According to Cauchy's residue theorem, the integral of $z^3 e^{iaz} / (z^4 + 4)$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{z^3 e^{iaz}}{z^4 + 4} dz = 2\pi i \left(\operatorname{Res}_{z=z_1} \frac{z^3 e^{iaz}}{z^4 + 4} + \operatorname{Res}_{z=z_2} \frac{z^3 e^{iaz}}{z^4 + 4} \right)$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{z^3 e^{iaz}}{z^4 + 4} dz + \int_{C_R} \frac{z^3 e^{iaz}}{z^4 + 4} dz = 2\pi i \left(\operatorname{Res}_{z=z_1} \frac{z^3 e^{iaz}}{z^4 + 4} + \operatorname{Res}_{z=z_2} \frac{z^3 e^{iaz}}{z^4 + 4} \right)$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: \quad z &= r, & r &= -R \rightarrow r = R \\ C_R: \quad z &= Re^{i\theta}, & \theta &= 0 \rightarrow \theta = \pi \end{aligned}$$

As a result,

$$\int_{-R}^R \frac{r^3 e^{iar}}{r^4 + 4} dr + \int_{C_R} \frac{z^3 e^{iaz}}{z^4 + 4} dz = 2\pi i \left(\operatorname{Res}_{z=z_1} \frac{z^3 e^{iaz}}{z^4 + 4} + \operatorname{Res}_{z=z_2} \frac{z^3 e^{iaz}}{z^4 + 4} \right).$$

Take the limit now as $R \rightarrow \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{r^3 e^{iar}}{r^4 + 4} dr = 2\pi i \left(\operatorname{Res}_{z=z_1} \frac{z^3 e^{iaz}}{z^4 + 4} + \operatorname{Res}_{z=z_2} \frac{z^3 e^{iaz}}{z^4 + 4} \right)$$

The denominator can be written as $z^4 + 4 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$. From this we see that the multiplicities of the $z - z_1$ and $z - z_2$ factors are both 1. The residues at $z = z_1$ and $z = z_2$ can then be calculated by

$$\begin{aligned} \operatorname{Res}_{z=z_1} \frac{z^3 e^{iaz}}{z^4 + 4} &= \phi_1(z_1) \\ \operatorname{Res}_{z=z_2} \frac{z^3 e^{iaz}}{z^4 + 4} &= \phi_2(z_2), \end{aligned}$$

where $\phi_1(z)$ and $\phi_2(z)$ are equal to $f(z)$ without the $z - z_1$ and $z - z_2$ factors, respectively.

$$\begin{aligned} \phi_1(z) &= \frac{z^3 e^{iaz}}{(z - z_2)(z - z_3)(z - z_4)} \Rightarrow \phi_1(z_1) = \frac{1}{4} e^{(-1+i)a} \\ \phi_2(z) &= \frac{z^3 e^{iaz}}{(z - z_1)(z - z_3)(z - z_4)} \Rightarrow \phi_2(z_2) = \frac{1}{4} e^{(-1-i)a} \end{aligned}$$

So then

$$\begin{aligned} \operatorname{Res}_{z=z_1} \frac{z^3 e^{iaz}}{z^4 + 4} &= \frac{1}{4} e^{(-1+i)a} \\ \operatorname{Res}_{z=z_2} \frac{z^3 e^{iaz}}{z^4 + 4} &= \frac{1}{4} e^{(-1-i)a} \end{aligned}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{r^3 e^{iar}}{r^4 + 4} dr &= 2\pi i \left[\frac{1}{4} e^{(-1+i)a} + \frac{1}{4} e^{(-1-i)a} \right] \\ \int_{-\infty}^{\infty} \frac{r^3 \cos ar + ir^3 \sin ar}{r^4 + 4} dr &= i\pi e^{-a} \left(\frac{e^{ia} + e^{-ia}}{2} \right) \\ \int_{-\infty}^{\infty} \frac{r^3 \cos ar}{r^4 + 4} dr + i \int_{-\infty}^{\infty} \frac{r^3 \sin ar}{r^4 + 4} dr &= i\pi e^{-a} \cos a. \end{aligned}$$

Match the real and imaginary parts of both sides.

$$\int_{-\infty}^{\infty} \frac{r^3 \cos ar}{r^4 + 4} dr = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{r^3 \sin ar}{r^4 + 4} dr = \pi e^{-a} \cos a$$

Therefore, changing the dummy integration variable to x ,

$$\boxed{\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4 + 4} dx = \pi e^{-a} \cos a.}$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \rightarrow \infty$. The parameterization of the semicircular arc in Fig. 93 is $z = Re^{i\theta}$, where θ goes from 0 to π .

$$\begin{aligned} \int_{C_R} \frac{z^3 e^{iaz}}{z^4 + 4} dz &= \int_0^\pi \frac{(Re^{i\theta})^3 e^{iaRe^{i\theta}}}{(Re^{i\theta})^4 + 4} (Rie^{i\theta} d\theta) \\ &= \int_0^\pi \frac{e^{iaR(\cos\theta + i\sin\theta)}}{R^4 e^{i4\theta} + 4} (R^4 i e^{i4\theta} d\theta) \\ &= \int_0^\pi \frac{e^{iaR\cos\theta} e^{-aR\sin\theta}}{R^4 e^{i4\theta} + 4} (R^4 i e^{i4\theta} d\theta) \end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned} \left| \int_{C_R} \frac{z^3 e^{iaz}}{z^4 + 4} dz \right| &= \left| \int_0^\pi \frac{e^{iaR\cos\theta} e^{-aR\sin\theta}}{R^4 e^{i4\theta} + 4} (R^4 i e^{i4\theta} d\theta) \right| \\ &\leq \int_0^\pi \left| \frac{e^{iaR\cos\theta} e^{-aR\sin\theta}}{R^4 e^{i4\theta} + 4} (R^4 i e^{i4\theta}) \right| d\theta \\ &= \int_0^\pi \frac{|e^{iaR\cos\theta}| |e^{-aR\sin\theta}|}{|R^4 e^{i4\theta} + 4|} |R^4 i e^{i4\theta}| d\theta \\ &= \int_0^\pi \frac{e^{-aR\sin\theta}}{|R^4 e^{i4\theta} + 4|} R^4 d\theta \\ &\leq \int_0^\pi \frac{e^{-aR\sin\theta}}{|R^4 e^{i4\theta}| - |4|} R^4 d\theta \\ &= \int_0^\pi \frac{e^{-aR\sin\theta}}{R^4 - 4} R^4 d\theta \\ &= \int_0^\pi \frac{e^{-aR\sin\theta}}{1 - \frac{4}{R^4}} d\theta \end{aligned}$$

Now take the limit of both sides as $R \rightarrow \infty$.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{z^3 e^{iaz}}{z^4 + 4} dz \right| \leq \lim_{R \rightarrow \infty} \int_0^\pi \frac{e^{-aR\sin\theta}}{1 - \frac{4}{R^4}} d\theta$$

Because the limits of integration do not depend on R , the limit may be brought inside the integral.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{z^3 e^{iaz}}{z^4 + 4} dz \right| \leq \int_0^\pi \lim_{R \rightarrow \infty} \frac{e^{-aR\sin\theta}}{1 - \frac{4}{R^4}} d\theta$$

Since θ lies between 0 and π , the sine of θ is positive. a is also positive. Thus, the exponent of e tends to $-\infty$, and the integral tends to zero.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{z^3 e^{iaz}}{z^4 + 4} dz \right| \leq 0$$

The magnitude of a number cannot be negative, and the only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{z^3 e^{iaz}}{z^4 + 4} dz \right| = 0 \quad \rightarrow \quad \lim_{R \rightarrow \infty} \int_{C_R} \frac{z^3 e^{iaz}}{z^4 + 4} dz = 0.$$