Exercise 7

Use residues to evaluate the improper integrals in Exercises 1 through 8.

$$\int_{-\infty}^{\infty} \frac{x \sin x \, dx}{(x^2+1)(x^2+4)}.$$

Solution

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{ze^{iz}}{(z^2+1)(z^2+4)},$$

and the contour in Fig. 93. Singularities occur where the denominator is equal to zero.

$$\begin{gathered} (z^2+1)(z^2+4) = 0 \\ z^2+1=0 \quad \text{or} \quad z^2+4=0 \\ z=\pm i \quad \text{or} \quad z = \pm 2i \end{gathered}$$

The singular points of interest to us are the ones that lie within the closed contour, $z = i$ and $z = 2i$.

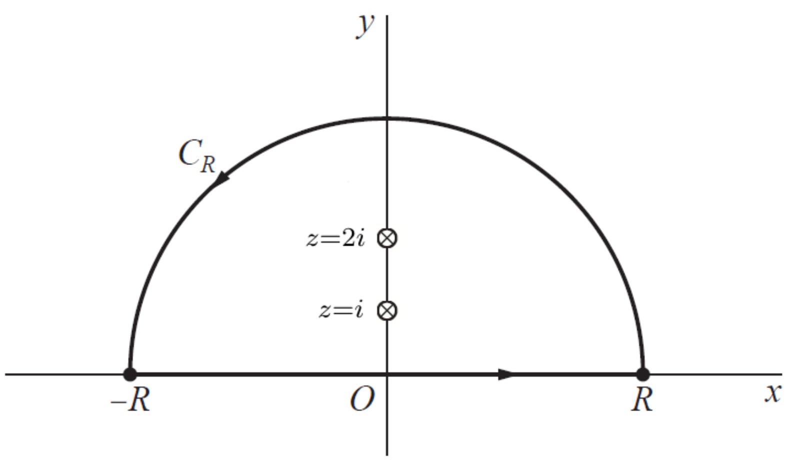

Figure 1: This is Fig. 93 with the singularities at $z = i$ and $z = 2i$ marked.

According to Cauchy's residue theorem, the integral of $ze^{iz}/[(z^2+1)(z^2+4)]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz = 2\pi i \left[\operatorname*{Res}_{z=i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} + \operatorname*{Res}_{z=2i} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\right]$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz + \int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz$$
$$= 2\pi i \left[\operatorname*{Res}_{z=i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} + \operatorname*{Res}_{z=2i} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\right]$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: &\quad z = r, & r = -R \quad &\to \quad r = R \\ C_R: &\quad z = Re^{i\theta}, & \theta = 0 \quad &\to \quad \theta = \pi \end{aligned}$$

As a result,

$$\begin{aligned} \int_{-R}^{R} \frac{re^{ir}}{(r^2+1)(r^2+4)}\,dr + \int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz \\ = 2\pi i \left[\operatorname*{Res}_{z=i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} + \operatorname*{Res}_{z=2i} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\right]. \end{aligned}$$

Take the limit now as $R \to \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{re^{ir}}{(r^2+1)(r^2+4)}\,dr = 2\pi i \left[\operatorname*{Res}_{z=i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} + \operatorname*{Res}_{z=2i} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\right]$$

The denominator can be written as $(z^2+1)(z^2+4) = (z+i)(z-i)(z+2i)(z-2i)$. From this we see that the multiplicities of the $z-i$ and $z-2i$ factors are both 1. The residues at $z = i$ and $z = 2i$ can then be calculated by

$$\begin{aligned} \operatorname*{Res}_{z=i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} &= \phi_1(i) \\ \operatorname*{Res}_{z=2i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} &= \phi_2(2i), \end{aligned}$$

where $\phi_1(z)$ and $\phi_2(z)$ are equal to $f(z)$ without the $z-i$ and $z-2i$ factors, respectively.

$$\begin{aligned} \phi_1(z) &= \frac{ze^{iz}}{(z+i)(z+2i)(z-2i)} \quad \Rightarrow \quad \phi_1(i) = \frac{1}{6}e^{-1} \\ \phi_2(z) &= \frac{ze^{iz}}{(z+i)(z-i)(z+2i)} \quad \Rightarrow \quad \phi_2(2i) = -\frac{1}{6}e^{-2} \end{aligned}$$

So then

$$\begin{aligned} \operatorname*{Res}_{z=i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} &= \frac{1}{6e} \\ \operatorname*{Res}_{z=2i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} &= -\frac{1}{6e^2} \end{aligned}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{re^{ir}}{(r^2+1)(r^2+4)}\,dr &= 2\pi i \left(\frac{1}{6e} - \frac{1}{6e^2}\right) \\ \int_{-\infty}^{\infty} \frac{r\cos r + ir\sin r}{(r^2+1)(r^2+4)}\,dr &= \frac{\pi i}{3}\left(\frac{1}{e} - \frac{1}{e^2}\right) \\ \int_{-\infty}^{\infty} \frac{r\cos r}{(r^2+1)(r^2+4)}\,dr + i\int_{-\infty}^{\infty} \frac{r\sin r}{(r^2+1)(r^2+4)}\,dr &= \frac{i\pi}{3}\left(\frac{e-1}{e^2}\right). \end{aligned}$$

Match the real and imaginary parts of both sides.

$$\int_{-\infty}^{\infty} \frac{r\cos r}{(r^2+1)(r^2+4)}\,dr = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{r\sin r}{(r^2+1)(r^2+4)}\,dr = \frac{\pi}{3}\left(\frac{e-1}{e^2}\right)$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\int_{-\infty}^{\infty} \frac{x\sin x\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}\left(\frac{e-1}{e^2}\right).}$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the semicircular arc in Fig. 93 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned}
\int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz &= \int_0^\pi \frac{Re^{i\theta}e^{iRe^{i\theta}}}{[(Re^{i\theta})^2+1][(Re^{i\theta})^2+4]}(Rie^{i\theta}\,d\theta) \\
&= \int_0^\pi \frac{e^{iR(\cos\theta+i\sin\theta)}}{(R^2e^{i2\theta}+1)(R^2e^{i2\theta}+4)}(R^2ie^{i2\theta}\,d\theta) \\
&= \int_0^\pi \frac{e^{iR\cos\theta}e^{-R\sin\theta}}{(R^2e^{i2\theta}+1)(R^2e^{i2\theta}+4)}(R^2ie^{i2\theta}\,d\theta)
\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}
\left|\int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz\right| &= \left|\int_0^\pi \frac{e^{iR\cos\theta}e^{-R\sin\theta}}{(R^2e^{i2\theta}+1)(R^2e^{i2\theta}+4)}(R^2ie^{i2\theta}\,d\theta)\right| \\
&\le \int_0^\pi \left|\frac{e^{iR\cos\theta}e^{-R\sin\theta}}{(R^2e^{i2\theta}+1)(R^2e^{i2\theta}+4)}(R^2ie^{i2\theta})\right| d\theta \\
&= \int_0^\pi \frac{\left|e^{iR\cos\theta}\right|\left|e^{-R\sin\theta}\right|}{|R^2e^{i2\theta}+1|\,|R^2e^{i2\theta}+4|}\left|R^2ie^{i2\theta}\right| d\theta \\
&= \int_0^\pi \frac{e^{-R\sin\theta}}{|R^2e^{i2\theta}+1|\,|R^2e^{i2\theta}+4|}R^2\,d\theta \\
&\le \int_0^\pi \frac{e^{-R\sin\theta}}{(|R^2e^{i2\theta}|-|1|)(|R^2e^{i2\theta}|-|4|)}R^2\,d\theta \\
&= \int_0^\pi \frac{e^{-R\sin\theta}}{(R^2-1)(R^2-4)}R^2\,d\theta \\
&= \int_0^\pi \frac{e^{-R\sin\theta}}{\left(1-\frac{1}{R^2}\right)\left(1-\frac{4}{R^2}\right)}\frac{d\theta}{R^2}
\end{aligned}$$

Now take the limit of both sides as $R \to \infty$.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz\right| \le \lim_{R\to\infty}\int_0^\pi \frac{e^{-R\sin\theta}}{\left(1-\frac{1}{R^2}\right)\left(1-\frac{4}{R^2}\right)}\frac{d\theta}{R^2}$$

Because the limits of integration do not depend on R, the limit may be brought inside the integral.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz\right| \le \int_0^\pi \lim_{R\to\infty}\frac{e^{-R\sin\theta}}{\left(1-\frac{1}{R^2}\right)\left(1-\frac{4}{R^2}\right)}\frac{d\theta}{R^2}$$

Since θ lies between 0 and π, the sine of θ is positive. Thus, the exponent of e tends to $-\infty$, and the integral tends to zero.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz\right| \le 0$$

The magnitude of a number cannot be negative, and the only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz\right| = 0 \quad\to\quad \lim_{R\to\infty}\int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz = 0.$$